Chapter 2 Geometry 1: Formal geometry: Triangles and polygons

Learning objectives

By the end of this chapter, the students should be able to:

1. Recall and prove elementary theorems in plane geometry:
   - The angle sum of a triangle is 180°.
   - The exterior angle of a triangle is equal to the sum of the opposite interior angles.
   - The angle sum of an *n*-sided convex polygon is $(2n - 4)$ right angles.
2. Recall and apply the conditions for triangles to be congruent:
   - Two sides and the included angle.
   - Two angles and a corresponding side.
   - Three sides.
   - Right angle, hypotenuse and side.
3. Use the elementary theorems and properties of congruent triangles to prove theorems and riders.
4. Identify properties of parallelograms and related quadrilaterals, and isosceles and equilateral triangles.
5. Use and apply the equal intercept theorem.
6. Apply theorems and riders to solve geometrical problems.

Teaching and learning materials

Students: Copy of textbook, exercise book and writing materials, paper, scissors or blade, geometrical instruments (especially protractors).
Teacher: A copy of the textbook, cardboard, paper scissors; chalkboard instruments (especially a protractor).

Glossary of terms

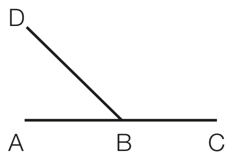
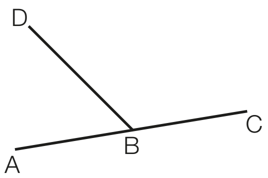
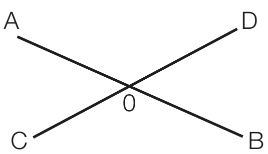
Rider is a problem based on a certain theorem or theorems. We say the problem rides on this/these specific theorem(s).
Re-entrant polygon (or a **concave polygon**) is a polygon that contains one or two reflex angles.

Teaching notes

- When you start doing geometry you could briefly mention the history of geometry.
  - Mention that the father of plane geometry is regarded as Euclid. He lived more than 3 centuries before the birth of Christ and was of Greek decent, but was born in Alexandria in Egypt. He also taught Mathematics there.
  - Euclid's book called "The Elements" consists of 13 volumes of which only one contains plane geometry.
  - Euclid did not create this geometry. Most of the geometry originated from earlier mathematicians or was practically used in navigation or surveying the land.
  - Euclid's main achievement is that he presented this geometry in a logically coherent framework. His work on plane geometry is one of the most influential works in mathematics, because it served as the main textbook for teaching geometry until late in the 19^{th} century/early 20^{th} century.
  - The geometry that is taught in schools today is merely an adaptation of how Euclid taught it. Some people also call plane geometry Euclidian Geometry.
- The logic used in Geometry is based on the principle that we assume certain facts, like the fact that the angles around a point add up to 360° or the fact that the sum of the angles on a straight line is equal to 180°.
  - Then we use these assumptions, called axioms, to prove theorems like the sum of the angles of a triangle is equal to 180°, and so on.

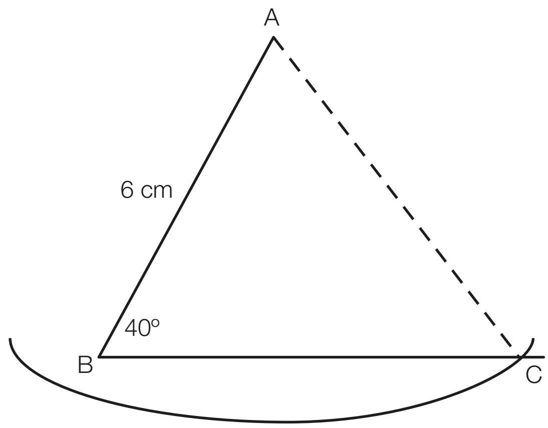
- These theorems are then used to solve problems in geometry or to prove riders.
- It is obvious that, if our assumptions are not true, that the whole logical system would collapse.

- When teaching the proof of the theorems, emphasise that students learn these proofs, because then they could be sure of those marks at least when answering a geometry question paper.
- When you do the examples, emphasise that students must give a reason for each statement they make if the statement is the result of a theorem or an axiom.
  - To help them you could give them summaries of the theorems and suggestions of the reasons they can use.
  - Below are suggestions of how these summaries could look:

Sketch	Theorem/Axiom	Reason you must give, if you use this Theorem/ Axiom
	If ABC is a straight line, $\angle ABD + \angle DBC = 180°$.	Sum $\angle$'s on a str. line = 180°
	If $\angle ABD + \angle DBC = 180°$, then AB and BC lie in a straight line.	The sum of adjacent angles = 180°
	If two straight lines AB and CD intersect at O, $\angle AOD = \angle BOC$ and $\angle AOC = \angle DOB$.	Vert. opp. $\angle$'s =

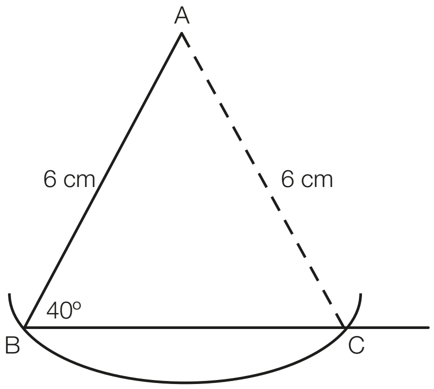
- Teach students to write out the solution of a geometrical rider as if they are explaining it to somebody else who does not understand easily.
  - So every step of the reasoning as well as the reasons for statements must be written down.
  - You can emphasise this when you ask the students to help you to write out the examples.
- **Congruency**
  - When you start your lesson about congruent triangles, you may mention that a triangle has 6 elements, namely 3 angles and 3 sides.
    - If we want to construct a triangle, however, we need only three of the six elements.
    - Then you can go through all the combinations of these three elements to determine which of them will have the result that the whole class would always get identical triangles if they use that particular combination to construct a triangle.
    - It would be a good idea if you could illustrate how the triangles are constructed so that the class can see that no other triangle is possible.
  - When you do the ambiguous case, you can illustrate why and when you can get 1 triangle, 2 triangles or no triangles as follows:
  - Tell the class that if you have a $\triangle ABC$, with $\angle B = 40°$ and AB = 6 cm, for example, that depending on the length of AC, you could get 1 or 2 triangles or no triangle:
    - To find C, you measure the required length on a ruler with your compass. Then you put the sharp, metal point of the compass on A and chop off the required length through the line by drawing an arc with the pencil of your compass.
    - Illustrate this by drawing a line BC on the board, measuring $\angle B = 40°$ and drawing a line from B through the 40° point.
    - Then you chop off a length that represents the length of BA.
    - After this, you stretch your compass to an arc wider/longer or equal or shorter than BA and draw the arc from A.

a) If AC > AB, it will look something like this:



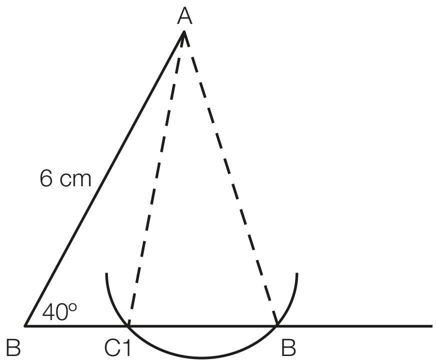
The arc drawn will intersect the line in one point only. So only one 1 triangle is possible.

b) If AC = AB, it will look something like this:



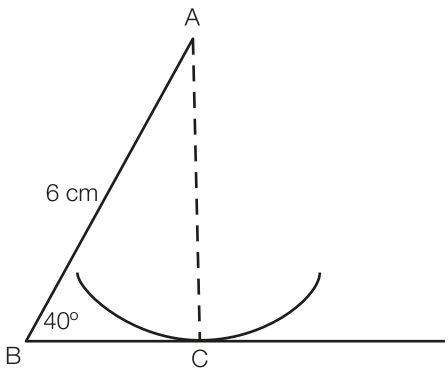
The arc drawn will intersect the line in one point only. So only one 1 triangle is possible.

c) If AC < 6 cm (and not so short that the arc will not intersect the base line), it will look something like this:

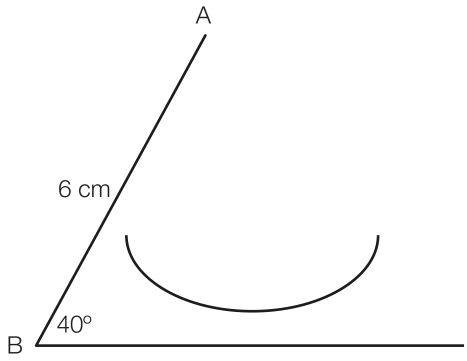


The arc drawn will intersect the line in two places. So, two triangles are possible. ($\triangle ABC_1$ and $\triangle ABC_2$)

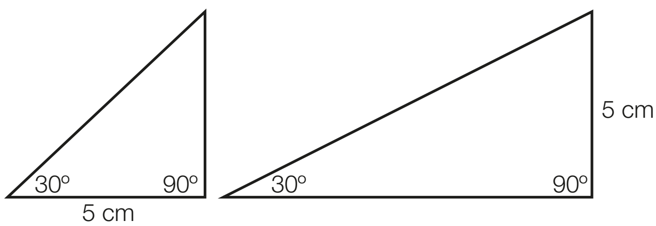
d) If AC is just long enough, the arc will just touch the line through B and only one triangle is possible. In this case, $\angle ACB = 90°$. That is where the fourth case of congruency comes from.



e) If AC is too short, no triangle is possible:



- In congruency problems, the lengths of sides are not given. So, if we do not know the length of the side opposite the given angle and the length of the side adjacent to that angle, we cannot say with certainty whether the triangles are congruent, because it could be any of the cases illustrated above.
- When you do the case of congruency where two angles and a side is used to construct a triangle, illustrate on the chalk board why the given side must be in the same position for the triangles to be congruent.
  - Draw any triangle with the given side between the two given angles and another triangle with the given side opposite one of the given angles, for example.
  - Below is a possible result.



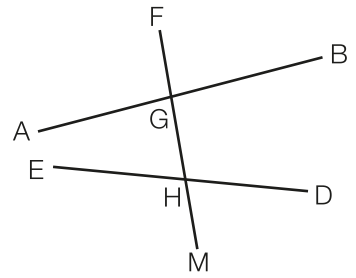
- When you do parallelograms and related quadrilaterals, it is important that you realise that students can be on different levels of understanding geometry. It is useful to know the Van Hiele levels of understanding geometry, especially if you want your students to be able to define a rectangle, or a square or a rhombus.
  - At level 0, a student can just recognise the quadrilateral.
  - At level 1, students recognise the properties of the quadrilateral. They would, for example, realise that a square has equal sides and that all its angles are right angles.

- At level 2, students can see that all squares are also rectangles and they can also write down definitions of figures, but they are unable to write down formal geometrical proofs.
- At level 3, they start to understand the meaning of deduction from simple proofs and can construct simple geometrical proofs.

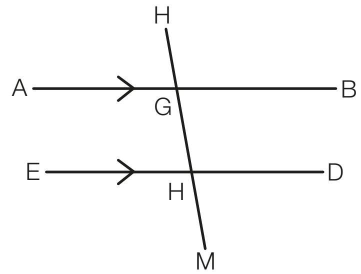
- When students have to prove that a quadrilateral with one pair of opposite sides equal and parallel, is a parallelogram, for example, they have to prove that both pairs of opposite sides of the quadrilateral are parallel (See Exercise 2d).
  - In other words, they have to go back to how the parallelogram was defined.
  - We could, of course, use any of the properties of the parallelogram (except the fact that a diagonal bisects its area) as the definition of the parallelogram and prove the other properties from that definition.
- Remember to tell students that a **good definition** must satisfy the following two properties:
  - It must be **economical**, which means that it is a short as possible while still very clear.
  - It must be **unique**, which means that there must be no confusion of what it defines.
- When you set geometry questions for a test or an examination, always make sure that your solution is the shortest possible path to follow in order to solve the problem.
  - In that way, you make sure that the problem does not count too many marks.
  - You have to, however, accept any answers from your students that are logical and correctly set out, even if it is very long.

Areas of difficulty and common mistakes

- Students do not write reasons for their statements. Make a summary of reasons for them as suggested above and insist on reasons.
- Students tend to leave out steps of their reasoning. Tell them that they are actually explaining the solution to somebody else and want to convince this person of their solution.
- When students state that alternate or corresponding angles are equal or that co-interior angles are supplementary, they omit to say that the lines through which the transversal passes are parallel. In the figure below:



  - ∠AGF and ∠EHG are corresponding angles and ∠AGH and ∠GHD are alternate angles, but these corresponding and alternate angles are not equal, because AB is not parallel to ED.
  - Similarly: ∠BGH and ∠DHG are co-interior angles, but they are not supplementary, because AB is not parallel to ED.
  - So, not all alternate angles and not all corresponding angles are equal and neither are all co-interior angles supplementary. The lines through which the transversal passes, must be parallel to each other.
  - So, it is essential that students say the following, for example:



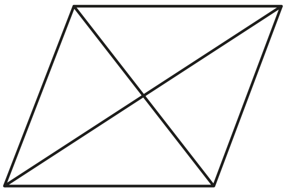
    $\angle AGF = \angle EHG$ (corresponding $\angle$s, AB || ED)
    Or
    $\angle AGH = \angle MHD$ (alternate $\angle$s, AB || ED)
    Or
    $\angle AGH + \angle EHG = 180°$ (co-interior $\angle$s, AB || ED)
- Students find it very difficult to define the different kinds of quadrilaterals.
  - You could give them the following summary and each time emphasise that the figure is a special parallelogram or rectangle or rhombus, which has all these properties in common with the parallelogram or rectangle or rhombus, but has special properties.

- Then let them try to define a rhombus, rectangle or square themselves.
- An example of a summary that you can use is shown below:

Parallelograms

1. Both pairs of opposite sides parallel (definition).
2. Both pairs of opposite angles equal.
3. Diagonals bisect each other.
4. One pair of opposite sides is equal and parallel.

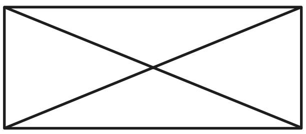
Rhombuses



1. Both pairs of opposite sides parallel.
2. Both pairs of opposite angles equal.
3. Diagonals bisect each other.
4. One pair of opposite sides is equal and parallel.

Special properties
1. All sides are equal.
2. Diagonals are perpendicular on each other.
3. Diagonals bisect the angles.

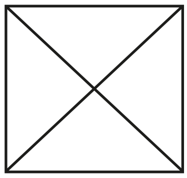
Rectangles



1. Both pairs of opposite sides parallel.
2. Both pairs of opposite angles equal.
3. Diagonals bisect each other.
4. One pair of opposite sides is equal and parallel.

Special properties
1. All the angles are equal to 90°.
2. Diagonals are equal.

Squares



1. Both pairs of opposite sides parallel.
2. Both pairs of opposite angles equal.
3. Diagonals bisect each other.
4. One pair of opposite sides is equal and parallel.

Special properties
1. All the angles are equal to 90°.
2. Diagonals are equal.
3. Diagonals bisect the angles.
4. Diagonals are perpendicular on each other.
5. All the sides are equal.

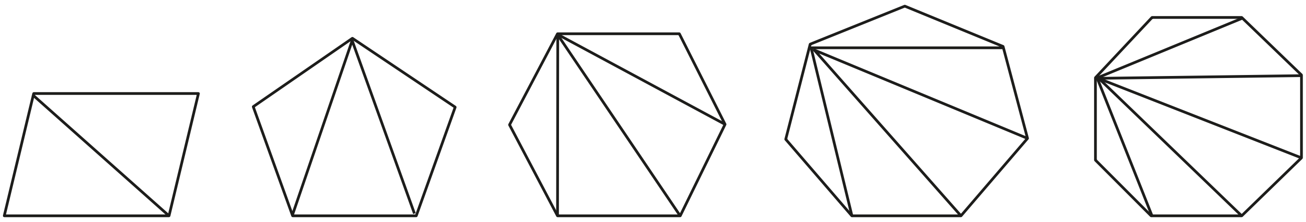
- Students tend to write the letters of two congruent triangles in the wrong order.
  - Emphasise that they have to write the letters according to the corresponding sides and angles that are equal.
  - Let them trace the letters according to the equal parts with their fingers and say the letters out loud.
- Students tend to say that two triangles are congruent if AAS of one triangle are equal to AAS of the other triangle when the equal sides are not in the same position. If necessary, let them construct a pair of triangles where the equal side is not in the same position.
- Students find it difficult to write out a proof or a how they calculate something in a geometrical figure.
  - The only cure for this is practise and explanation and more practise and explanation.
  - It also helps to let the student first verbally explain the solution or proof to you and then write out the whole proof or calculation like an essay.
  - After this, the students can "translate" their essay in the formal form of a proof as illustrated in the textbook.

Supplementary worked examples

- There is an alternate method of proving Theorem 3:

Let the class copy and complete this table, by using the polygons below the table:

Name of figure	Number of sides	Number of $\triangle$s	Number of right angles	Total degrees
Triangle				
Quadrilateral				
Pentagon				
Hexagon				
Heptagon				
Octagon				
n-gon				



Quadrilatera Pentagon Hexagon Heptagon Octagon

Name of figure	Number of sides	Number of $\triangle$s	Number of right angles	Total degrees
Triangle	3	1	2	$2 \times 90° = 180°$
Quadrilateral	4	2	4	$4 \times 90° = 360°$
Pentagon	5	3	6	$6 \times 90° = 540°$
Hexagon	6	4	8	$8 \times 90° = 720°$
Heptagon	7	5	10	$10 \times 90° = 900°$
Octagon	8	6	12	$12 \times 90° = 1\,080°$
n-gon	n	$n - 2$	$2(n - 2) = 2n - 4$	$2 \times 90°(n - 2) = 180°(n - 2)$

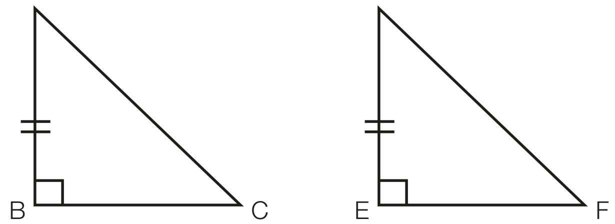
Answer

There is an alternate method of proving Theorem 4:
Each interior angle + each exterior angle = 180°.
The sum of n interior angles + the sum of n exterior angles = $180°n$.
$180°(n - 2)$ + the sum of n exterior angles = $180°n$.

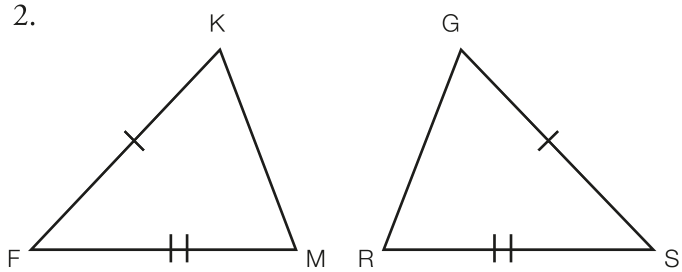
$$\begin{aligned}\text{The sum of } n \text{ exterior angles} &= 180°n - 180°(n - 2)\\ &= 180°n - 180n + 360°\\ &= 360°\end{aligned}$$

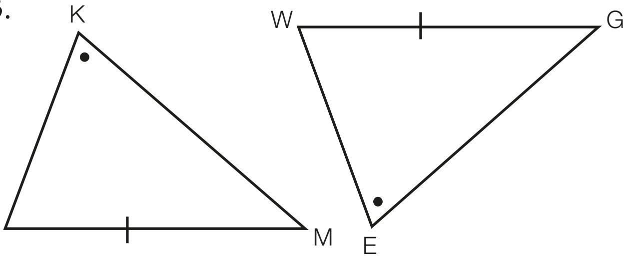
- You could also add these kinds of problems to Exercise 2b:
  - Two triangles with two elements equal, and the student has to add another pair of elements for the two triangles to be congruent.
  - Here are some examples:
    - In each of these pairs of triangles, two pairs of equal elements are marked.
    - In each case, write down another pair of equal elements for the triangles to be congruent.
    - Give the congruency case, which you used and also give all the possibilities without repeating any congruency test you already used:

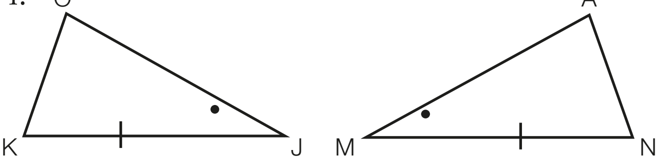
1.

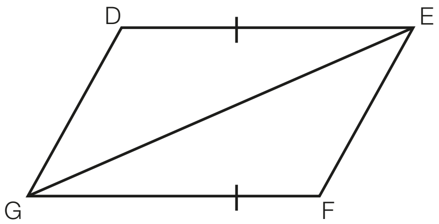


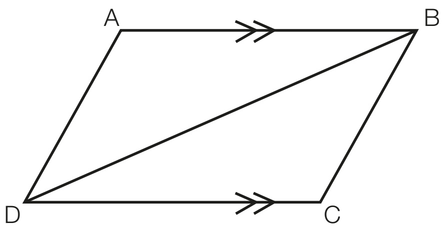
2.



3.


4.


5.


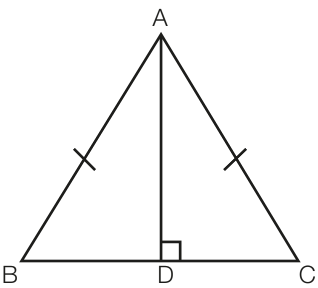
6.


Solutions

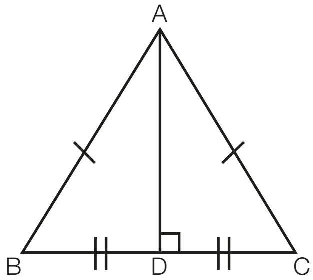
1. |AC| = |DF| (RHS), |BC| = |EF| (SAS), ∠A = ∠D (AAS = AA corr S)
2. |KM| = |GR| (SSS), ∠F = ∠R (SAS)
3. ∠M = ∠W (AAS = AA corr S)
4. |OJ| = |MA| (SAS), ∠O = ∠A (AAS = AA corr S)
5. ∠DEG = ∠FGE (SAS), |DG| = |EF| (SSS)
6. |AB| = |DC| (SAS), ∠A = ∠C (AAS = AA corr S)

There are two other constructions that can be used to prove Theorem 5:

Construction: Draw AD ⊥ BC so that D is on BC



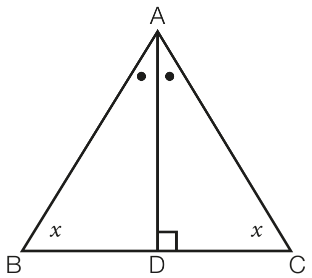
Proof: In △ABD and △ACD:
(1) |AB| = |AC| (given)
(2) |AD| = |AD| (same side)
(3) ∠BDA = ∠CDA (construction)
∴ △ABD ≡ △ACD (RHS)
∴ ∠B = ∠C (corr angles in △s ABD and ACD)
Construction: Draw AD with D on BC so that BD = DC



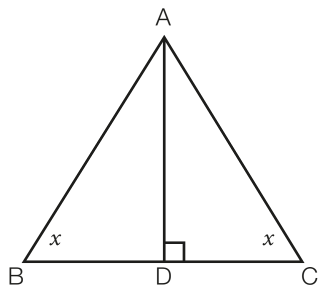
Proof: In △ABD and △ACD:
1) |AB| = |AC| (given)
2) |BD| = |DC| (construction)
3) |AD| = |AD| (same side)
∴ △ABD ≡ △ACD (SSS)
∴ ∠B = ∠C (corr angles in △s ABD and ACD)

Theorem 5 states: The base angles of an isosceles triangle are equal. The converse is also true, however:

If two angles of triangle are equal, the triangle is isosceles.
Given: △ABC with ∠B = ∠C.
To prove: AB = AC.
Construction: Draw the bisector of ∠A to meet BC at D.



Proof: : In △ABD and △ACD:
1) ∠B = ∠C (given)
2) |AD| = |AD| (same side)
3) ∠BAD = ∠DAC (construction)
∴ △ABD ≡ △ACD (AAS = AA corr S)
∴ AB = AC (corr sides in △'s ABD and ACD)
Or:
Given: △ABC with ∠B = ∠C.
To prove: AB = AC.
Construction: Draw AD ⊥ BC with D on BC.



Proof: In $\triangle ABD$ and $\triangle ACD$:

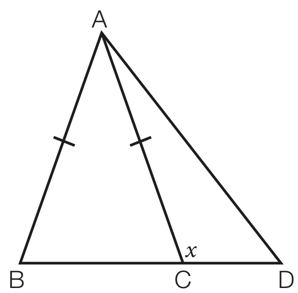
1) $\angle B = \angle C$	(given)
2) $\lvert AD \rvert = \lvert AD \rvert$	(same side)
3) $\angle BDA = \angle ADC$	(construction)
$\therefore \triangle ABD \equiv \triangle ACD$	(AAS = AA corr S)

$\therefore$ AB = AC (corr sides in $\triangle$s ABD and ACD)

If you draw AD so that BD = DC, you would get SSA which is the ambiguous case and not a congruency case. So, there are only two alternate proofs for this theorem.

An alternate problem for questions 9 and 17 of Exercise 2c is the following. (It is a rather difficult problem and can be given to the more talented students in your class.):

In the figure, AB = AC = BD. Prove, with reasons, that $\angle ACD = 2\angle ADB$.



Let $\angle ACD = x$.

$\therefore \angle ACB = 180° - x$	(sum $\angle$s on a str. line = 180°)
$= \angle B$	(AB = AC)
$\angle BAD = \angle ADB$	(AB = BD)
$= \frac{180° - (180° - x)}{2} = \frac{x}{2}$	

$\therefore \angle ACD = x = 2\angle ADB$

Class activity

You could let the class work in groups to draw representations of family trees of how all the quadrilaterals are related.

They then have to write down why their family tree looks a certain way and they then must use their family tree to write down definitions for each of the quadrilaterals.

Below is an example of a representation: