N5 Mathematics

Student Book

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Pearson South Africa (Pty) Ltd
4th floor, Auto Atlantic Building, Corner of Hertzog Boulevard and Heerengracht, Cape Town, 8001

Offices in Johannesburg, Durban, East London, Polokwane, Bloemfontein, Rustenburg and Mbombela.

website: za.pearson.com

First published in 2022
ISBN (print) 9781485720775
ISBN (epdf) 9781485720959

Publisher: Lloyd Stuurman
Editor: Judith Elliott
Managing editor: Monique Maartens
Proofreader: Christine de Nobrega
Artwork: Marco Casati
Book design: Pearson Media Hub
Cover design: Pearson Media Hub
Cover artwork: Paul Aparicio/Shutterstock
Typesetting: Marco Casati
Printed by xxxx printers, (city)

Acknowledgements:
Pg 1. AlineBehrendt Photography. Shutterstock, pg 14. Peter Titmuss. Shutterstock, pg 17. 7505811966. Shutterstock, pg 23. Richard Whitcombe. Shutterstock, pg 33. Igor Nikushin. Shutterstock, pg 36a RomarioIen. Shutterstock, pg 36b frantic00. Shutterstock, pg 41. Dr. Victor Wong. Shutterstock, pg 53. Felix Lipov. Shutterstock, pg57. Ehrman Photographic. Shutterstock, pg 71. Olivier Le Moal. Shutterstock, pg 75. supergenijalac. Shutterstock, pg 81. Frame Stock Footage. Shutterstock, pg 83. Nomad_Soul. Shutterstock, pg 84. maradon 333. Shutterstock, pg 85. Pretty Vectors. Shutterstock, pg 86 Smit. Shutterstock, pg 88. Yulia Furman. Shutterstock, pg 92. Authentic travel. Shutterstock, pg 128. BearFotos. Shutterstock, pg 145. superjoseph. Shutterstock, pg 157. Tupungato. Shutterstock, pg 169 Art Directors & TRIP . Alamy Stock Photo, pg 170. Tim Roberts Photography. Shutterstock, pg 172. TonelsonProductions. Shutterstock, pg 175. Mariusz Niedzwiedzki. Shutterstock, pg 176a. Ken Schulze. Shutterstock, pg 176b. Engineer studio. Shutterstock, pg 179. Studio 8. Pearson Education Ltd, pg 184. radist. 123rf.com, pg 185. Rawpixel.com. Shutterstock, pg 188. gresei. Shutterstock, pg 189. BlueRingMedia. Shutterstock, pg 191. ThePhotoFab. Shutterstock, pg 194. Dotted Yeti. Shutterstock.

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Module

1 Limits and continuity

What is covered?

In this module, we revise different types of limits and apply limit laws. We then look at the applications of L'Hospital's rule to indeterminate functions. This rule finds the limit after differentiating the numerator and denominator.

That is, if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

We also investigate the conditions for continuity and determine whether a function is continuous or discontinuous at a specific point.

Subject outcomes

After studying this module, you should be able to:

- understand limits and apply limit laws
- state the conditions for continuity and determine whether a function is continuous or discontinuous at a specific point.

Figure 1.1 Worm's eye view of giant Douglas fir trees in Perthshire, Scotland

Maximum height

Every tree has its maximum height, depending on the species it belongs to. The maximum height for a Douglas fir tree is 138 metres. This is the height limit for one of the world's tallest trees.

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Unit 1: Limits

LEARNING OUTCOMES

- Understand limits and use limit laws.
- Apply L'Hospital's rule (differentiate the numerator and denominator) to indeterminate functions.

Introduction

Limits are central to understanding the behaviour of functions; the limit concept is the core of all calculus studies. Mathematical modelling and the use of calculus in engineering fields rely on your understanding of limits. In simple terms, a limit is used to describe a function when its input approaches a specific value (near a point but not necessarily at that point). We say that it is arbitrarily close to a point but it is not dependent on the value of the function at that point.

1. Limits and limit laws

Let $f(x)$ be a function defined on an interval that contains $x = a$, but may exclude a. We write:

$$\lim_{x \to a} f(x) = L$$

where L is the limit of $f(x)$ as x approaches a.
For this definition to hold, we need the left-hand limit and the right-hand limit of $f(x)$ to both be equal to L as $x \to a$ from left and from right.

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

For example, given $f(x) = 2x^3$.
As x approaches 1 from both sides, $f(x)$ approaches 2.
So,

$$\lim_{x \to 1} 2x^3 = 2$$

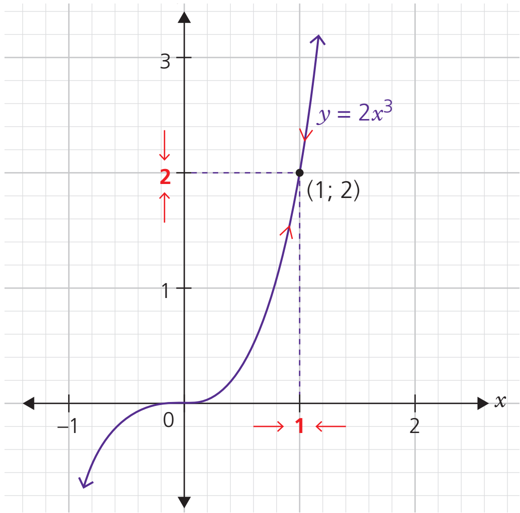

Figure 1.2 The graph of $f(x) = 2x^3$

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In another example, given

$$f(x) = \begin{cases} 2x^3, & x < 1 \\ x, & x \geq 1 \end{cases}$$

As x approaches 1, $f(x)$ approaches 2 from the left, but 1 from the right.
So,

$$\lim_{x \to 1^-} f(x) = 2 \text{ and } \lim_{x \to 1^+} f(x) = 1$$

In this case, $\lim\limits_{x \to 1} 2x^3$ does not exist, because the left-hand and the right-hand limits are not the same.

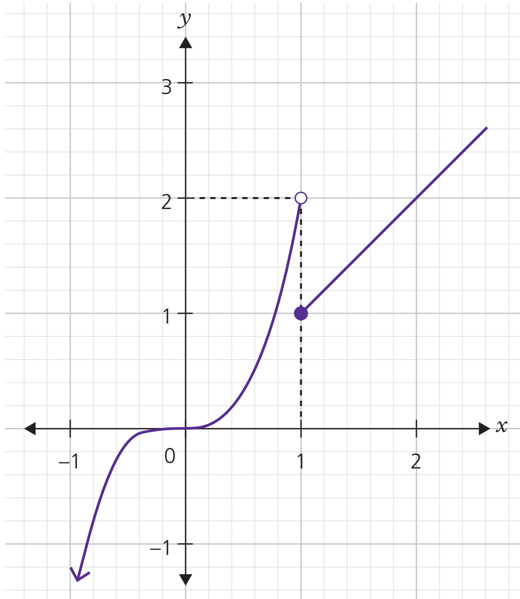

Figure 1.3 The graph of $f(x) = \begin{cases} 2x^3, & x < 1 \\ x, & x \geq 1 \end{cases}$

Infinite limits

Let $f(x)$ be a function defined on both sides of $x = a$, but possibly excluding a.
Then

$$\lim_{x \to a} f(x) = \infty$$

means that as the value of x gets close to a, the values of $f(x)$ get infinitely large.
On the graph of $f(x)$ the line $x = a$ is called a **vertical asymptote**.
For example, in Figure 1.3, $x = -1$ is a vertical asymptote of f and we have the following limit:

Keyword
vertical asymptote: the vertical line $x = a$ where at least one of the limits of $f(x)$ is $\pm\infty$

$$\lim_{x \to -1^-} f(x) = \infty \text{ and } \lim_{x \to -1^+} f(x) = \infty, \text{ so } \lim_{x \to -1} f(x) = \infty$$

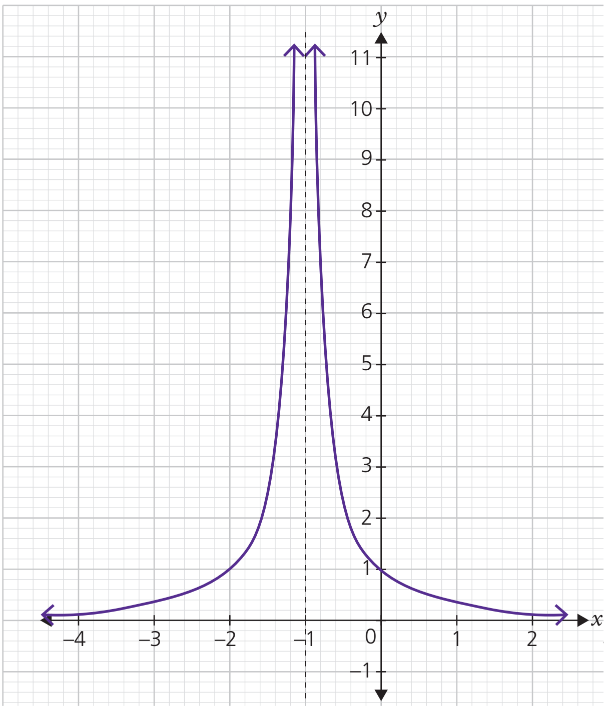

Figure 1.4 $x = -1$ is a vertical asymptote of f

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In the example in Figure 1.4, $x = -1$ is a vertical asymptote of f and we have: $\lim_{x \to -1^-} f(x) = -\infty$ and $\lim_{x \to -1^+} f(x) = +\infty$, so $\lim_{x \to -1} f(x)$ doesn't exist, because the corresponding one-sided limits are not equal.

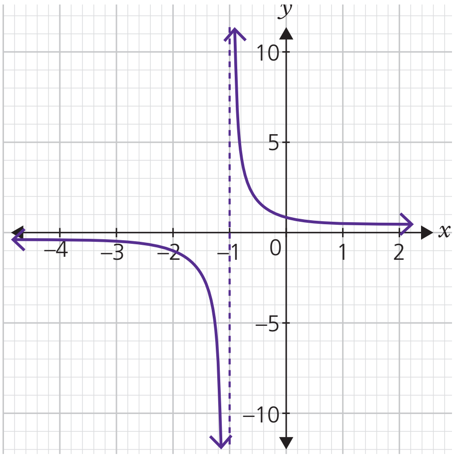

Figure 1.5 $x = -1$ is a vertical asymptote of f

Limit laws

In N4 Mathematics you used the limit laws to calculate limits. You used the **direct substitution property** and learnt that sometimes you need to manipulate the algebraic expression first before finding the limit. You also used simplification, rationalisation and division by the highest power of x.

> **Keyword**
>
> **direct substitution property:** if f is a polynomial or a rational function, then $\lim_{x \to a} f(x) = f(a)$.
>
> ($f(a)$ must be defined. In other words, a is in the domain of f.)

LIMIT LAWS

Given two functions, $f(x)$ and $g(x)$, such that the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then the following limit laws apply:

Law 1: $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

Law 2: $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

Law 3: $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$ where c is a constant

Law 4: $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

Law 5: $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

Law 6: $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

Law 7: $\lim_{x \to a} k = k$ where k is a constant, $k \in \mathbb{R}$

Law 8: $\lim_{x \to a} x = a$

Law 9: $\lim_{x \to a} x^n = a^n$ where n is a positive integer

Law 10: $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

Law 11: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

> **Remember**
>
> Special limits
>
> - $\lim_{x \to \pm\infty} \frac{1}{x} = 0$
> - $\lim_{x \to \pm\infty} \frac{k}{x^n} = 0$
> - $\lim_{x \to \infty} e^x = \infty$
> - $\lim_{x \to -\infty} e^x = 0$
> - $\lim_{x \to \infty} \ln x = \infty$
> - $\lim_{x \to 0^+} \ln x = -\infty$ (right-hand limit)

Worked example 1.1

Given the functions $f(x) = 3x^2 - 16x - 12$ and $g(x) = 3x + 2$.
Using limit laws and direct substitution, evaluate the following limits:

1. $\lim_{x\to 0}[3f(x)]$

2. $\lim_{x\to -3}[f(x) + 2g(x)]$

3. $\lim_{x\to 1}[3f(x) - g(x)]$

4. $\lim_{x\to -1}[f(x)g(x)]$

5. $\lim_{x\to -4}\sqrt{f(x)}$

Solutions

1. $\lim_{x\to 0}[3f(x)] = 3\lim_{x\to 0}[f(x)]$ Law 3

$= 3\lim_{x\to 0}[3x^2 - 16x - 12]$

$= 3(0 - 0 - 12) = 3(-12) = -36$ direct substitution

2. $\lim_{x\to -3}[f(x) + 2g(x)] = \lim_{x\to -3} f(x) + \lim_{x\to -3} 2g(x)$ Law 1

$= \lim_{x\to -3} f(x) + 2\lim_{x\to -3} g(x)$ Law 3

$= \lim_{x\to -3}[3x^2 - 16x - 12] + 2\lim_{x\to -3}[3x + 2]$

$= [3(-3)^2 - 16(-3) - 12] + 2[3(-3) + 2]$ direct substitution

$= 27 + 48 - 12 - 18 + 4 = 49$

3. $\lim_{x\to 1}[3f(x) - g(x)] = \lim_{x\to 1} 3f(x) - \lim_{x\to 1} g(x)$ Law 2

$= 3\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)$ Law 3

$= 3\lim_{x\to 1}[3x^2 - 16x - 12] + \lim_{x\to 1}[3x + 2]$

$= 3(3 - 16 - 12) + (3 + 2)$ direct substitution

$= 9 - 48 - 36 + 5 = -70$

4. $\lim_{x\to -1}[f(x)g(x)] = \lim_{x\to -1} f(x) \cdot \lim_{x\to -1} g(x)$ Law 4

$= \lim_{x\to -1}[3x^2 - 16x - 12] \cdot \lim_{x\to -1}[3x + 2]$

$= [3(-1)^2 - 16(-1) - 12] \cdot [3(-1) + 2]$ direct substitution

$= (3 + 16 - 12)(-1) = -7$

In the limits above, we also used Laws 7, 8 and 9.
Try to identify places where these laws were used.

5. $\lim_{x\to -4}\sqrt{f(x)} = \lim_{x\to -4}\sqrt{3x^2 - 16x - 12}$

$= \sqrt{\lim_{x\to -4}[3x^2 - 16x - 12]}$ Law 11

$= \sqrt{[3(-4)^2 - 16(-4) - 12]} = \sqrt{100} = 10$ direct substitution

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Worked example 1.2

Find $\lim_{x \to -\frac{2}{3}} \left[\frac{3x^2 - 16x - 12}{3x + 2}\right]$

Solution

Apply Law 5: $\dfrac{\lim_{x \to -\frac{2}{3}} [3x^2 - 16x - 12]}{\lim_{x \to -\frac{2}{3}} [3x + 2]}$

But $\lim_{x \to -\frac{2}{3}} [3x + 2] = \left[3\left(-\frac{2}{3}\right) + 2\right] = -2 + 2 = 0$

So, we need another way of finding this limit.

$\dfrac{3x^2 - 16x - 12}{3x + 2} = \dfrac{(3x + 2)(x - 6)}{3x + 2} = x - 6$ factorise the numerator and cancel out the common factor

Now, $\lim_{x \to -\frac{2}{3}} [x - 6] = -\frac{2}{3} - 6 = -6\frac{2}{3}$

So, $\lim_{x \to -\frac{2}{3}} \left[\frac{3x^2 - 16x - 12}{3x + 2}\right] = -6\frac{2}{3}$

Worked example 1.3

Evaluate $\lim_{x \to \infty} \dfrac{3x^8 - 2x^3}{6x^4 - 7x^5}$

Solution

In this limit, both the numerator and the denominator approach ∞ as $x \to \infty$.
One way of evaluating this limit is to divide each term by the highest power of x in the denominator. In this example, this is x^5.

So, $\dfrac{3x^8 - 2x^3}{6x^4 - 7x^5} = \dfrac{\frac{3x^8}{x^5} - \frac{2x^3}{x^5}}{\frac{6x^4}{x^5} - \frac{7x^5}{x^5}} = \dfrac{3x^3 - \frac{2}{x^2}}{\frac{6}{x} - 7}$

Now we can use the limit laws to evaluate the given limit:

$$\lim_{x \to \infty} \frac{3x^8 - 2x^3}{6x^4 - 7x^5} = \lim_{x \to \infty} \frac{3x^3 - \frac{2}{x^2}}{\frac{6}{x} - 7} = \frac{\lim_{x \to \infty} 3x^3 - \lim_{x \to \infty}\left(\frac{2}{x^2}\right)}{\lim_{x \to \infty} \frac{6}{x} - \lim_{x \to \infty} 7}$$

$$= \frac{3\lim_{x \to \infty} x^3 - 2\lim_{x \to \infty}\left(\frac{1}{x^2}\right)}{6\lim_{x \to \infty} \frac{1}{x} - \lim_{x \to \infty} 7} = \frac{\infty - 0}{0 - 7} = \frac{\infty}{-7} = -\infty$$

Worked example 1.4

Find $\lim_{x \to 0} \dfrac{\sqrt{x^3 + 4} - 2}{x^3}$

Solution

In this example, direct substitution doesn't work: $\dfrac{\sqrt{0^3 + 4} - 2}{0} = \dfrac{\sqrt{4} - 2}{0} = \dfrac{0}{0}$
We can use another method to rationalise the numerator (because the square root is in the numerator).

$$\frac{\sqrt{x^3+4}-2}{x^3} = \frac{\left(\sqrt{x^3+4}-2\right)\left(\sqrt{x^3+4}+2\right)}{x^3\left(\sqrt{x^3+4}+2\right)} = \frac{\left(\sqrt{x^3+4}\right)^2-(2)^2}{x^3\left(\sqrt{x^3+4}+2\right)}$$

$$= \frac{x^3+4-4}{x^3\left(\sqrt{x^3+4}+2\right)} = \frac{x^3}{x^3\left(\sqrt{x^3+4}+2\right)} = \frac{1}{\sqrt{x^3+4}+2}$$

Now we can find the limit:

$$\lim_{x\to 0}\frac{\sqrt{x^3+4}-2}{x^3} = \lim_{x\to 0}\frac{1}{\sqrt{x^3+4}+2} = \frac{1}{\sqrt{0+4}+2} = \frac{1}{4}$$

ACTIVITY 1.1 (PAIR) Finding limits using limit laws

1. Calculate the limits.

a) $\lim_{x\to -\frac{\pi}{2}}(\csc x - \sin x)$

b) $\lim_{x\to 0}\frac{xe^x}{x^2e^x - x}$

c) $\lim_{x\to\infty}\left(-\frac{2}{\ln x} - \frac{1}{x}\right)$

d) $\lim_{x\to -1}\frac{x^4-2x^3+6x-3}{x^3-2x^2+7x-1}$

2. Given the functions $f(x) = x^2 - 6x + 9$ and $g(x) = x - 3$.
Evaluate the following limits:

a) $\lim_{x\to 0}\left[f(x)^2\right]$

b) $\lim_{x\to -5}\left[f(x) - \sqrt[3]{g(x)}\right]$

c) $\lim_{x\to -2}[-3f(x) + g(x)]$

d) $\lim_{x\to 0}[f(x)g(x)]$

e) $\lim_{x\to 3}\frac{\sqrt{f(x)}}{1-g(x)}$

3. Determine the following limits:

a) $\lim_{x\to 0}\left(\frac{\arctan 4x}{\arccos 3x}\right)$

b) $\lim_{x\to 1}(e^x \ln x)$

c) $\lim_{x\to 0}\frac{4^x+2^x}{x^2-1}$

4. Find $\lim_{x\to -\infty}\left[\frac{3x^2-12x}{3x}\right]$

5. Evaluate:

a) $\lim_{x\to\infty}\left[\frac{2x^9+x^6}{-2x^4+11x^7}\right]$

b) $\lim_{x\to -\infty}\left[\frac{7x^2-3x+1}{-3x^2+x+5}\right]$

6. Find the limits.

a) $\lim_{x\to 0}\frac{3-\sqrt{9-2x}}{x}$

b) $\lim_{x\to 0}\frac{-3x}{\sqrt{x+1}-1}$

7. Given $\ln y^2 = \lim_{x\to 0}\frac{e^x-1}{x^2+1}$, calculate the numerical value of:

a) $\ln y^2$

b) y

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2. L'Hospital's Rule

In the previous section we used different methods for finding limits that give $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when directly substituting the x-value. For example,

$\lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2}$ would give $\frac{0}{0}$ if we substitute $x = 2$ directly, and

$\lim\limits_{x \to \infty} \frac{2x^3 + 1}{1 - x^3}$ would give $\frac{\infty}{-\infty}$ if we substitute ∞ into the expression without manipulating it first.

These types of limits are called **indeterminate**.

Keyword

indeterminate: Not defined; indeterminate limits of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, 0^0, $\infty - \infty$

Some of these limits are not easy to manipulate algebraically, such as

$\lim\limits_{x \to 0} \frac{\sin x}{x^2}$ or $\lim\limits_{x \to \infty} \frac{\ln x}{x}$.

L'Hospital's Rule is used to evaluate limits of the form $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or both $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$.

These limits are called indeterminate forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

The other types of indeterminate limits that we will be working with are:
$0 \times \infty$, 0^0, $\infty - \infty$.
These need to be manipulated in order to obtain the quotient form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hospital's rule gives us a way to deal with limits that are indeterminate. This rule tells us that to find the limit, we need to differentiate the numerator and the denominator and take the limit of the resulting fraction.
That is, if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

where a is any real number or $\pm\infty$.

It is important to check that the given functions satisfy these conditions:

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

$$\text{or} \quad \lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

Summary of differentiation rules

General rules

Power rule $y = x^n$	$\frac{dy}{dx} = nx^{n-1}$
Constant rule $y = k$	$\frac{dy}{dx} = 0$
Multiple rule	$\frac{d}{dx}[cf(x)] = cf'(x)$
Sum rule	$[f(x) + g(x)]' = f'(x) + g'(x)$
Difference rule	$[f(x) - g(x)]' = f'(x) - g'(x)$
Product rule	$[f(x) \times g(x)]' = f'(x)g(x) + f(x)g'(x)$
Quotient rule	$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Trigonometric functions

sine function: $f(x) = \sin x$	$f'(x) = \cos x$
cosine function: $f(x) = \cos x$	$f'(x) = -\sin x$
tangent function: $f(x) = \tan x$	$f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$
cotangent function: $f(x) = \cot x$	$f'(x) = -\operatorname{cosec}^2 x = -\frac{1}{\sin^2 x}$
secant function: $f(x) = \sec x$	$f'(x) = \sec x \tan x$
cosecant function: $f(x) = \operatorname{cosec} x$	$f'(x) = -\operatorname{cosec} x \cot x$

Exponential and logarithmic functions

Exponential function $f(x) = e^x$	$f'(x) = e^x$
Exponential function $f(x) = a^x$	$f'(x) = a^x \ln a$
Natural logarithm function $f(x) = \ln\lvert x\rvert$	$f'(x) = \frac{1}{x}$
Logarithmic function $f(x) = \log_a x$	$f'(x) = \frac{1}{x \ln a}$

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Worked example 1.5

Use L'Hospital's rule to find the following limits.

1. $\lim_{x\to 3}\frac{x^2-9}{x-3}$

2. $\lim_{x\to 1}\frac{x^3-2x^2-5x+6}{2x^3-2x^2+3x-3}$

3. $\lim_{x\to 0}\frac{\sin x}{x}$

4. $\lim_{x\to\infty}\frac{\ln x}{x}$

Solutions

1. First find the limits of the numerator and denominator separately.

$\lim_{x\to 3}(x^2-9)=3^2-9=0$ and $\lim_{x\to 3}(x-3)=3-3=0$

This is an indeterminate form $\frac{0}{0}$ and we can apply L'Hospital's rule.

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x^2-9)'}{(x-3)'}=\lim_{x\to 3}\frac{2x}{1}=6$$

2. First find the limits of the numerator and denominator separately.

$$\frac{\lim_{x\to 1}(x^3-2x^2-5x+6)}{\lim_{x\to 1}(2x^3-2x^2+3x-3)}=\frac{(1)^3-2(1)^2-5(1)+6}{2(1)^3-2(1)^2+3(1)-3}=\frac{1-2-5+6}{2-2+3-3}=\frac{0}{0}$$

So, this is an indeterminate limit of the form $\frac{0}{0}$.

Use L'Hospital's rule:

$$\lim_{x\to 1}\frac{(x^3-2x^2-5x+6)'}{(2x^3-2x^2+3x-3)'}=\lim_{x\to 1}\frac{3x^2-4x-5}{6x^2-4x+3}=\frac{3(1)^2-4(1)-5}{6(1)^2-4(1)+3}=\frac{3-4-5}{6-4+3}=-\frac{6}{5}$$

3. Using direct substitution we get: $\lim_{x\to 0}\frac{\sin x}{x}=\frac{\sin 0}{0}=\frac{0}{0}$

Use L'Hospital's rule:

$$\lim_{x\to 0}\frac{(\sin x)'}{x'}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{1}=1$$

4. $\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{(\ln x)'}{x'}=\lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{1}=\lim_{x\to\infty}\frac{1}{x}=0$

Sometimes, you will need to apply L'Hospital's rule more than once to find the limit.

Worked example 1.6

Find the limit:

$$\lim_{x\to 0}\frac{\sin x-x}{\frac{1}{2}x^2}$$

Solution

First find the limits of the numerator and denominator separately.

$$\lim_{x\to 0}\frac{\sin x-x}{\frac{1}{2}x^2}=\frac{\lim_{x\to 0}(\sin x-x)}{\lim_{x\to 0}\left(\frac{1}{2}x^2\right)}=\frac{0-0}{0}=\frac{0}{0}$$ indeterminate form

Use L'Hospital's rule:

$$\lim_{x\to 0}\frac{\sin x-x}{\frac{1}{2}x^2}=\lim_{x\to 0}\frac{(\sin x-x)'}{\left(\frac{1}{2}x^2\right)'}=\lim_{x\to 0}\frac{\cos x-1}{\frac{1}{2}\times 2x}=\frac{1-1}{0}=\frac{0}{0}$$ indeterminate form

$$\lim_{x\to 0}\frac{\cos x-1}{x}=\lim_{x\to 0}\frac{(\cos x-1)'}{x'}=\lim_{x\to 0}\frac{-\sin x}{1}=\frac{-0}{1}=0$$

So, $\lim_{x\to 0}\frac{\sin x-x}{\frac{1}{2}x^2}=0$.

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Worked example 1.7

Calculate the following limits:

1. $\lim\limits_{x\to\infty}(e^x - x^2)$

2. $\lim\limits_{x\to\infty}\left(\frac{1}{e^x}\cdot 2^x\right)$

3. $\lim\limits_{x\to 1^+}(x^2 - 2x + 1)^{x-1}$

Solutions

1. $\lim\limits_{x\to\infty}(e^x - x^2) = \lim\limits_{x\to\infty}(e^x) - \lim\limits_{x\to\infty}(x^2) = \infty - \infty$

First manipulate the limit.

Multiply and divide by e^x:

$$\lim_{x\to\infty}(e^x - x^2) = \lim_{x\to\infty}(e^x - x^2)\times\frac{e^x}{e^x} = \lim_{x\to\infty}e^x\left(1 - \frac{x^2}{e^x}\right)$$

The limit of product is the product of limits and $\lim e^x = \infty$

Use L'Hospital's rule twice: $\lim\limits_{x\to\infty}\left(\frac{x^2}{e^x}\right) = \frac{\lim\limits_{x\to\infty}(x^2)'}{\lim\limits_{x\to\infty}(e^x)'} = \frac{\lim\limits_{x\to\infty}(2x)'}{\lim\limits_{x\to\infty}(e^x)'} = \frac{\lim\limits_{x\to\infty}2}{\lim\limits_{x\to\infty}e^x} = 0$

So, $\lim\limits_{x\to\infty}(e^x - x^2) = \lim\limits_{x\to\infty}(e^x) = \infty$

2. $\lim\limits_{x\to\infty}\left(\frac{1}{e^x}\cdot 2^x\right) = \lim\limits_{x\to\infty}\left(\frac{1}{e^x}\right)\cdot\lim\limits_{x\to\infty}(2^x) = 0\cdot\infty$

First manipulate the limit.

$\frac{1}{e^x}\cdot 2^x = \frac{2^x}{e^x} = \left(\frac{2}{e}\right)^x$

Now, take the limit:

$\lim\limits_{x\to\infty}\left(\frac{1}{e^x}\cdot 2^x\right) = \lim\limits_{x\to\infty}\left(\frac{2}{e}\right)^x = 0$

3. $\lim\limits_{x\to 1^+}(x^2 - 2x + 1)^{x-1} = (1^2 - 2(1) + 1)^{1-1} = 0^0$

First manipulate the limit.

$y = (x^2 - 2x + 1)^{x-1}$

$y = (x - 1)^{2(x-1)}$

$x^2 - 2x + 1 = (x - 1)^2$

To bring the power $2(x - 1)$ down, take the logarithm of both sides:

$\ln y = 2(x - 1)\ln(x - 1)$

$y = e^{2(x-1)\ln(x-1)}$

Now, take the limit:

$$\lim_{x\to 1^+}(x^2 - 2x + 1)^{x-1} = \lim_{x\to 1^+}e^{2(x-1)\ln(x-1)} = e^{\lim\limits_{x\to 1^+}2(x-1)\ln(x-1)}$$

$$\lim_{x\to 1^+}2(x - 1)\ln(x - 1) = 2\lim_{x\to 1^+}\frac{\ln(x - 1)}{\frac{1}{x - 1}}$$

change to a fraction

$$= 2\lim_{x\to 1^+}\frac{(\ln(x - 1))'}{\left(\frac{1}{x - 1}\right)'} = 2\lim_{x\to 1^+}\frac{\frac{1}{x - 1}}{-\left(\frac{1}{x - 1}\right)^2} = 2\lim_{x\to 1^+}(1 - x) = 0$$

Finally,

$$\lim_{x\to 1^+}(x^2 - 2x + 1)^{x-1} = e^{\lim\limits_{x\to 1^+}2(x-1)\ln(x-1)} = e^0 = 1$$

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ACTIVITY 1.2 SINGLE — Finding limits using L'Hospital's rule

1. Verify whether the given limits are indeterminate forms. Find the limits if they exist.

a) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

b) $\lim_{x \to -1} \frac{3x^2 - 1}{1 + x}$

c) $\lim_{x \to -4} \frac{x^2 - 16}{x + 4}$

d) $\lim_{x \to 1} \frac{x^2 - 1}{\ln x}$

e) $\lim_{x \to 1} \frac{x^3 - x^2 + 4x - 4}{x - 1}$

f) $\lim_{x \to 3} \frac{x^3 - 3x^2 - 2x + 6}{x - 3}$

g) $\lim_{x \to 0} \frac{\tan x - x}{x^3 - 3}$

h) $\lim_{x \to -1} \frac{1 + x^5}{1 + x^{15}}$

i) $\lim_{x \to \infty} \frac{\sin x}{\cos x + 1}$

j) $\lim_{x \to \infty} \frac{e^x}{2x}$

k) $\lim_{x \to \infty} \left(\frac{x + 1}{x}\right)^2$

l) $\lim_{x \to -\infty} \frac{1 - x^2}{x}$

2. Calculate $\lim_{x \to 1^+} \frac{\ln x}{\sqrt{x - 1}}$

3. Calculate $\lim_{x \to \infty} \frac{e^x}{2x^2}$

4. Find the following limits:

a) $\lim_{x \to 0} \frac{\sin x - x}{x^2}$

b) $\lim_{x \to \pi} \frac{-\sin x}{(x - \pi)^2}$

c) $\lim_{x \to 0} \frac{x - \sin x}{\tan x}$

5. Find $\lim_{x \to \infty} \frac{ax^2 + bx}{cx + d}$

6. Calculate the following limits:

a) $\lim_{x \to \infty} \left(\frac{e^x}{6} - x\right)$

b) $\lim_{x \to \infty} 2xe^{-x}$

c) $\lim_{x \to 0^+} x \ln x$

d) $\lim_{x \to 0^-} (x^2 - 7x)^x$

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