N4 Mathematics

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Module
1

Determinants

What is covered?

Many situations in science and engineering involve working with a system of equations, which is a collection of equations that share the same variables. The solution to a system is the set of values that satisfy all the equations. This module will discuss 2×2 and 3×3 systems of linear equations. Some of these systems have solutions while others do not.

Determinants are numbers that predict whether a system will have a solution, and they are also used to find solutions. You will learn to calculate determinants using minors and cofactors, and also use Cramer's rule to solve for variables in a system of linear equations.

Subject outcomes

After studying this module, you should be able to:

Unit 1

- Use a determinant to represent linear equations in two variables.
- Calculate second order determinants.
- Use a determinant to describe the nature of the solution to a linear system.

Unit 2

- Use a determinant to represent linear equations in three variables.
- Calculate minors and cofactors for a third order determinant.
- Calculate third order determinants using cofactor expansion along a row.

Unit 3

- Apply Cramer's rule to solve a system of linear equations.

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Unit 1: Second order determinants

LEARNING OUTCOMES

- Use a determinant to represent linear equations in two variables.
- Calculate second order determinants.
- Use a determinant to describe the nature of the solution to a linear system.

Introduction

A 2 × 2 linear system has two equations in two variables: $ax + by = m$ and $cx + dy = n$. A system like this can have different types of solutions.

- This unit discusses how to find the second order determinant for a given 2 × 2 linear system.
- The value of the determinant predicts what type of solution a system will have.

1. Solutions to a 2 × 2 linear system

Remember

$ax + by = c$ can be written as $y = -\frac{a}{b}x + \frac{c}{b}$, which is a line with a gradient of $-\frac{a}{b}$.

Linear equations in two variables represent straight lines, and gradient values determine whether or not two lines will intersect each other.

ACTIVITY 1.1 GROUP **Investigate the nature of solutions**

Figures 1.1a), 1.1b) and 1.1c) show three 2 × 2 systems of linear equations.

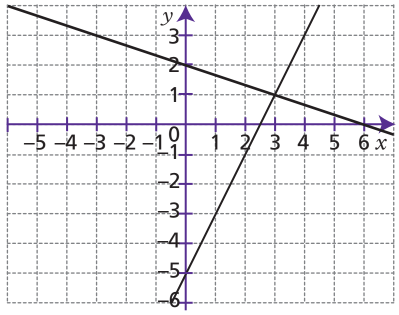

Figure 1.1a) Lines defined by $2x - y = 5$ and $x + 3y = 6$

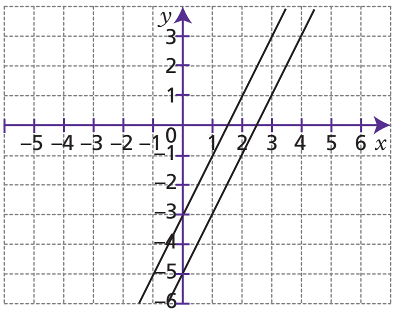

Figure 1.1b) Lines defined by $2x - y = 5$ and $2x - y = 3$

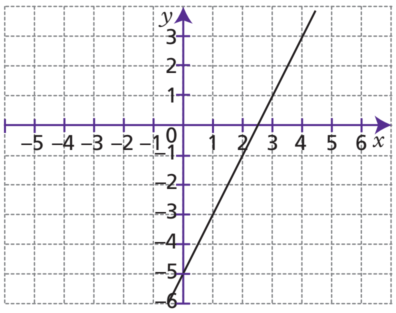

Figure 1.1c) Lines defined by $2x - y = 5$ and $4x - 2y = 10$

1. Give the gradients of the lines in each of the three systems.
2. a) Which pair of lines has no intersection?
   b) Which pair has one point of intersection?
   c) Which pair of lines intersects at an infinite number of points?
3. What do you notice about the gradients of the lines that intersect each other exactly once?

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In the investigation in Activity 1.1, only the **system of equations** in Figure 1.1a) has a unique **solution**: the lines are not parallel and the point (3;1) lies on both lines, so $x = 3$, $y = 1$ is the only solution to both equations. Lines that have the same gradient are parallel to each other, so they will either not intersect at all, giving no solution as in Figure 1.1b), or they will be the same line with an infinite number of points in common as in Figure 1.1c).

In general, a 2 × 2 linear system will only have a unique solution if the gradients of the two lines are not equal.

Keywords

system of equations a set of equations

solution makes an equation or a set of equations true

ACTIVITY 1.2 PAIR **Predict the nature of solutions**

1. Use gradients to decide if there will be a unique solution to each system of equations:
   - **a)** $3x–y = 4$ and $2y + x = 3$
   - **b)** $x + y + 4 = 0$ and $2x = 6 - 2y$
2. A 2 × 2 system consists of the equations $ax + y = m$ and $x + dy = n$. Use gradients to prove that the system only has a unique solution if $ad \neq 1$.

2. The determinant of a 2 × 2 linear system

A 2 × 2 system $ax + by = m$ and $cx + dy = n$ only has a unique solution if the gradients are not equal:

- $-\frac{a}{b} \neq -\frac{c}{b}$
- $ad \neq bc$
- $ad - bc \neq 0$

The important expression $ad - bc$, which combines the coefficients of the linear system, is called a 2 × 2 (or second order) **determinant**, shown by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Keyword

determinant (of a square matrix) a quantity obtained by the addition of products of the elements of a square matrix according to a given rule

Definition:

Given the system $ax + by = m$ and $cx + dy = n$, the 2 × 2 determinant is defined by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. The system will have a unique solution if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

Worked example 1.1

For each system of equations in Questions 1 and 2 below, evaluate the 2 × 2 determinant that contains the coefficients. Determine what type of solution the system will have.

1. $3x - y = 6$
   $4x + 2y = 1$
2. $y = 3x - 6$
   $6x - 1 = 2y$
3. • Tablet A contains 400 g of potassium and 50 grams of zinc.
   • Tablet B contains 300 g of potassium and 80 grams of zinc.
   How many ways are there to combine the two tablets to provide 3 000 g of potassium and 4 000 g of zinc?

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Solution

1. Place the coefficients into a determinant (in the correct order) and evaluate:
   $\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix}$
   $= (3)(2) - (-1)(4)$
   $= 6 + 4$
   $= 10$
   The determinant is not zero, so the system will have a unique solution: there is only one value for x and one value for y that will satisfy both equations in the system.
2. Write the equations in the form $ax + by = c$:
   $3x - y = 6$
   $6x - 2y = 1$
   The determinant is given by:
   $\begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix}$
   $= (3)(-2) - (-1)(6)$
   $= -6 + 6$
   $= 0$
   The determinant is zero, so the system will not have a unique solution: there will either be no solution or an infinite number of solutions.
3. Let x represent the number of A tablets, let y represent the number of B tablets, and form a system of two equations:
   potassium: $400x + 300y = 3\ 000$
   zinc: $50x + 80y = 4\ 000$
   The nature of the solution is shown by the determinant that contains the coefficients of the system:
   $\begin{vmatrix} 400 & 300 \\ 50 & 80 \end{vmatrix}$
   $= (400)(80) - (50)(300)$
   $= 32\ 000 - 15\ 000$
   $= 17\ 000$
   The determinant is not zero, so there will be a unique solution: there is only one way to combine Tablets A and B that will satisfy the given conditions.

ACTIVITY 1.3 SINGLE Calculate second order determinants

1. For each 2 × 2 system, evaluate the second order determinant that contains the coefficients. State whether the system has a unique solution.
   a) $3x - y = 6$
   $x - 2y = 2$
   b) $3x - 4y = 1$
   $6x - 8y = 6$
   c) $x + y = 2$
   $3x + y = 6$

2. Use a determinant to confirm that this system has a unique solution:
$5 + y = 2x$
$6 - x - 3y = 0$
Hint: First write each equation in the form $ax + by = c$.

3. Solve for x:
a) $\begin{vmatrix} x & 1 \\ 6 & -2 \end{vmatrix} = 4$
b) $\begin{vmatrix} 9 & x \\ x & 1 \end{vmatrix} = 0$

4. An electrical circuit must be set up with currents I_1 and I_2 so that $2I_1 + I_2 = 3$ and $I_1 - 2I_2 = 4$.
Calculate the determinant of the coefficients to find how many possible values there are for the currents I_1 and I_2 .

5. Two forces F_1 and F_2 will act at a single point, so that $3F_1 - 4 = 2F_2$ and $6 + F_2 = 1{,}5F_1$.
Use a determinant to decide if there is exactly one possible value for each of the forces.

Unit 2: Third order determinants

LEARNING OUTCOMES

- Use a determinant to represent linear equations in three variables.
- Calculate minors and cofactors for a third order determinant.
- Calculate third order determinants using cofactor expansion along a row.

Introduction

A 3 × 3 system of linear equations has three equations in three variables.

Determinants are useful for predicting the nature of solutions and for calculating solutions. This unit provides a method for finding the third order determinant of a 3 × 3 system by using numbers known as **cofactors** and **minors**.

Keywords

cofactor The cofactor of the entry a_{ij} in row i and column j of a determinant is found by (–1)i+ j × minor

minor (of an entry in a determinant) a number calculated by deleting one row and one column to find the reduced determinant

1. The minor of an entry

An entry in a determinant can be labelled as a_{ij}, where i is its row number and j is its column number.

Worked example 1.2

	Column 1 ↓	Column 2 ↓	Column 3 ↓
Row 1 →	1	(2)	3
Row 2 →	5	6	–1
Row 3 →	(4)	0	2

- The number '2' is in row 1 and in column 2, so it is the entry a_{12}.
- The number '4' is in row 3 and in column 1, so it is the entry a_{31}.

Each entry in a determinant is associated with a number called the minor. This number is calculated by deleting one row and one column to find the reduced determinant. The minor of an entry in a 3 × 3 determinant will be a 2 × 2 determinant.

Definition:

Each entry a_{ij} in a determinant has its own minor. The minor M_{ij} is the value of the smaller determinant that results from deleting row i and column j, which contain the entry.

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Worked example 1.3

You are given the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & -1 \\ 4 & 0 & 2 \end{vmatrix}$.

1. Find the minor of the entry 5.
2. Find the minor of the entry 6.
3. Which minor is this: $\begin{vmatrix} 5 & 6 \\ 4 & 0 \end{vmatrix}$?

Solutions

1. 5 is in row 2 and column 1, so its minor M_{21} is found by deleting row 2 and column 1:

   $M_{21} = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & -1 \\ 4 & 0 & 2 \end{vmatrix}$

   Evaluate the smaller 2 × 2 determinant:

   $$\text{minor of } 5 = M_{21} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = (2)(2) - (0)(3) = 4$$

2. Delete row 2 and column 2, which contain the entry 6:

   $M_{22} = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & -1 \\ 4 & 0 & 2 \end{vmatrix}$

   $$\text{minor of } 6 = M_{22} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (1)(2) - (4)(3) = -10$$

3. The given minor is formed by deleting row 1 and column 3 of the determinant, so the minor is M_{13}, which is the minor associated with the entry 3.

ACTIVITY 1.4 PAIR Calculate minors

1. For this determinant $\begin{vmatrix} 4 & 1 & 5 \\ -3 & 2 & -1 \\ -1 & 6 & -5 \end{vmatrix}$, find:
   - **a)** M_{12} (the minor of the entry in row 1 column 2).
   - **b)** The minor of the entry –3.
   - **c)** M_{31}.
2. You are given the determinant $\begin{vmatrix} 1 & 4 & 3 \\ 4 & x & -1 \\ 2 & 6 & -2 \end{vmatrix}$. Find x if $M_{13} = 10$.

2. The cofactor of an entry

The cofactor of an entry in a determinant is the minor with a positive or negative sign. The sign depends on which row and which column the entry is in.

Definition:

The cofactor of an entry in a determinant is found by $(-1)^{\text{row + column}} \times$ minor:
cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} \times M_{ij}$.

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Worked example 1.4

If the determinant $\begin{vmatrix} 1 & 3 & -1 \\ 4 & 5 & -1 \\ 6 & 2 & 0 \end{vmatrix}$ is given, find:

1. The minor and cofactor of 4.
2. C_{33}, the cofactor of 0.

Solutions

1. Delete row 2 and column 1: $\begin{vmatrix} 1 & 3 & -1 \\ 4 & 5 & -1 \\ 6 & 2 & 0 \end{vmatrix}$

   minor of 4 = $M_x = \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$
   $= (3)(0) - (2)(-1)$
   $= 0 + 2$
   $= 2$

   The cofactor is the minor with a positive or negative sign:
   cofactor of 4 = $C_{21} = (-1)^{\text{row + column}} \times M_{21}$
   $= (-1)\ 2 + 1 \times 2$
   $= (-1)(2)$
   $= -2$

2. $C_{33} = (-1)^{\text{row + column}} \times$ minor of 0
   $= (-1)^{3+3} \times M_{33}$
   $= +1 \times \begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix}$
   $= 1[(1)(5)-(4)(3)]$
   $= 5 - 12$
   $= -7$

ACTIVITY 1.5 SINGLE Calculate cofactors

1. You are given the determinant $\begin{vmatrix} 4 & 1 & -1 \\ 2 & 6 & 0 \\ 3 & 5 & -2 \end{vmatrix}$. Find the minors and cofactors of all nine entries.
2. Given $\begin{vmatrix} 3 & 4 & -1 \\ -2 & 0 & 5 \\ -1 & 8 & 2 \end{vmatrix}$, find C_{32} (the cofactor of 8).
3. Given $\begin{vmatrix} 4 & 1 & 1 \\ -2 & 3 & 5 \\ 1 & 6 & 7 \end{vmatrix}$, find C_{21}.
4. The determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is given.
   a) Find only the positive or negative sign of each entry's cofactor.
   b) Explain why $(-1)^{\text{row + column}}$ forms this alternating pattern of signs: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$.

After completing Activity 1.5, do you see that the cofactor signs $(-1)^{\text{row + column}}$ form an alternating sign pattern: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$.

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3. Calculate a third order determinant

For a given determinant, there is an interesting result when you multiply each entry in a row by its cofactor.

ACTIVITY 1.6 GROUP **Multiply entries by their cofactors**

For each of the questions below, use the determinant $\begin{vmatrix} 4 & 1 & -1 \\ 2 & 6 & 0 \\ 3 & 5 & -2 \end{vmatrix}$ (from Question 1 of Activity 1.5).

Table 1.1 shows the three entries in row 1, their cofactors and the product of each entry with its cofactor.

1. Complete column 5 of Table 1.1, to show that when each entry in row 1 is multiplied by its cofactor and the three products are added together, the result is –36.

Table 1.1 Combine entries and cofactors in row 1

	Entry	Cofactor	Entry × cofactor	Sum of the three products
row 1	$a_{11} = 4$	$C_{11} = -12$	$a_{11}C_{11} = -48$	$a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13}$ $= -48 + \ldots\ldots\ldots\ldots$ $= \ldots.$
	$a_{12} = 1$	$C_{12} = 4$	$a_{12}C_{12} = 4$	
	$a_{13} = -1$	$C_{13} = -8$	$a_{13}C_{13} = 8$	

2. Complete Table 1.2 for the entries and cofactors in row 2 of the given determinant.

Table 1.2 Combine entries and cofactors in row 2

	Entry	Cofactor	Entry × cofactor	Sum of the three products
row 2	$a_{21} = 2$	$C_{21} = -3$	$a_{21}C_{21} = \ldots$	$a_{21} \cdot C_{21} + a_{22} \cdot C_{22} + a_{23} \cdot C_{23}$ $= \ldots$
	$a_{22} = 6$	$C_{\ldots} = -5$	$\ldots = \ldots$	
	$\ldots$	$C_{\ldots} = -17$	$\ldots = \ldots$	

3. Complete Table 1.3 for the entries and cofactors in row 3 of the given determinant.

Table 1.3 Combine entries and cofactors in row 3

	Entry	Cofactor	Entry × cofactor	Sum of the three products
row 3	$a_{\ldots} = 3$	$C_{\ldots} = 6$	$\ldots = \ldots$	$a_{\ldots} \cdot C_{\ldots} + a_{\ldots} \cdot C_{\ldots} + a_{\ldots} \cdot C_{\ldots}$ $= \ldots$
	$a_{\ldots} = 5$	$C_{\ldots} = -2$	$\ldots = \ldots$	
	$\ldots$	$C_{\ldots} = 22$	$\ldots = \ldots$	

4. What do you notice about the final sum in each of the three tables?
5. **a)** Which row or column of the determinant contains the entries a_{11}, a_{21} and a_{31}?
   b) Use the cofactors in the above tables to evaluate $a_{11} \cdot C_{11} + a_{21} \cdot C_{21} + a_{31} \cdot C_{31}$ for the given determinant.
   c) What do you notice about your answer to 5b)?

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The investigation in Activity 1.6 shows an important result: the same value is found when each entry in one row (or column) of a determinant is multiplied by its cofactor and the resulting products are added together. This provides a method for finding a third order determinant, by expanding along a row and using cofactors.

Definition:

Find a third order determinant by cofactor expansion along row 1:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13}$$

$$= a \cdot [(-1)^{1+1} \cdot \text{minor}] + b \cdot [(-1)^{1+2}.\text{minor}] + c \cdot [(-1)^{1+3} \cdot \text{minor}]$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$

Worked example 1.5

1. Evaluate $\begin{vmatrix} 2 & 1 & -1 \\ 0 & 2 & 4 \\ 3 & -5 & 3 \end{vmatrix}$ by using cofactor expansion along row 1.
2. Evaluate $\begin{vmatrix} -3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{vmatrix}$.
3. Use a determinant to predict whether this system of equations will have a unique solution:
   $4x - 4y + z = 2$
   $x - z = 3$
   $2y + 4z = 3$

Solutions

1. Find the cofactor of each entry in row 1.
   The cofactor of 2 is $(-1)^{\text{row + column}} \times$ minor of 2:
   $C_{11} = (-1)^{1+1} \times M_{11}$
   $= +1 \times \begin{vmatrix} 2 & 4 \\ -5 & 3 \end{vmatrix}$
   $= (2)(3) - (-5)(4)$
   $= 6 + 20$
   $= 26$
   Cofactor of 1 is $(-1)^{\text{row + column}} \times$ minor of 1:
   $C_{12} = (-1)^{1+2} \times M_{12}$
   $= -1 \times \begin{vmatrix} 0 & 4 \\ 3 & 3 \end{vmatrix}$
   $= -1[(0)(3) - (3)(4)]$
   $= -(0 - 12)$
   $= 12$
   Cofactor of −1 is $(-1)^{\text{row + column}} \times$ minor of −1:
   $C_{13} = (-1)^{1+3} \times M_{13}$
   $= +1 \times \begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix}$
   $= (0)(-5) - (3)(2)$
   $= 0 - 6$
   $= -6$

To calculate the determinant, multiply each entry by its cofactor and add the results:

$$\begin{vmatrix} 2 & 1 & -1 \\ 0 & 2 & 4 \\ 3 & -5 & 3 \end{vmatrix}$$
$= a_{11}.C_{11} + a_{12}.C_{12} + a_{13}.C_{13}$
$= 2 \cdot \text{cofactor} + 1 \cdot \text{cofactor} + (-1) \cdot \text{cofactor}$
$= 2(26) + 1(12) + (-1)(-6)$
$= 70$

2. Expand along row 1, multiplying each entry by its cofactor, which contains the alternating sign and the minor.
Combine the three calculations.
(The first two steps are optional.)

$$\begin{vmatrix} -3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{vmatrix}$$
$= a_{11}.C_{11} + a_{12}.C_{12} + a_{13}.C_{13}$
$= -3\left[(-1)^{1+1}M_{11}\right] + 2\left[(-1)^{1+2}M_{12}\right] + 1\left[(-1)^{1+3}M_{13}\right]$
$= -3(+1)\begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} + 2(-1)\begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix} + 1\left(+1\right)\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix}$
$= -3(1-0) - 2(2 + 12) + 1(0-4)$
$= -3(1) - 2(14) + 1(-4)$
$= -3 - 28 - 4$
$= -35$

3. Write the coefficients of the equations in a determinant and evaluate by expanding along row 1:

$$\begin{vmatrix} 4 & -4 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 4 \end{vmatrix} = 4\begin{vmatrix} 0 & -1 \\ 2 & 4 \end{vmatrix} - 4(-1)\begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$$
$= 4(0 + 2) + 4(4-0) + 1(2-0)$
$= 4(2) + 4(4) + 2$
$= 26$

The determinant is non-zero, so the system will have a unique solution.

Note
When expanding a 3 × 3 determinant along a row, multiply the three parts in each term: an entry a_{ij}, a sign $(-1)^{i+j}$ and M_{ij} (a 2 × 2 determinant).

ACTIVITY 1.7 SINGLE Work with third order determinants

1. In Example 1 above, the cofactor expansion along row 1 showed that
$$\begin{vmatrix} 2 & 1 & -1 \\ 0 & 2 & 4 \\ 3 & -5 & 3 \end{vmatrix} = 70.$$
Evaluate the same determinant by using cofactor expansion along row 2.
(Multiply each entry in row 2 by its cofactor and add the three products.)

2. Use cofactor expansion along row 1 to calculate $\begin{vmatrix} 6 & 2 & -4 \\ 5 & 6 & -2 \\ 5 & 2 & -3 \end{vmatrix}$.

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Reminder

A non-zero determinant shows a unique solution to a system of equations.

3. The 3 × 3 linear system is given:

$x + 3y + 2z = 3$

$2x - y + 3z = 0$

$5y - z = -2$

a) Write the coefficients of the system in a 3 × 3 determinant, keeping the same order of equations and variables. (Represent any missing variable with a zero coefficient.)

b) Expand along row 1 to find the value of the determinant in 3a).

c) What can you conclude about the solution to this system of equations?

4. A 3 × 3 system is represented by this determinant that contains the coefficients:

$$\begin{vmatrix} 2-k & 0 & 0 \\ 2 & 3-k & 4 \\ 1 & 2 & 1-k \end{vmatrix}$$

a) Use cofactor expansion along row 1 to show that the determinant is equal to $(2-k)(k-5)(k+1)$.

b) Find the value(s) of k for which the system will have a unique solution.

5. Find p if $\begin{vmatrix} 3 & p & -2 \\ 0 & 2 & 6 \\ 7 & 4 & p \end{vmatrix} = 4$.

6. Figure 1.2 shows three vectors that form a prism:
$\boldsymbol{a} = (1;\ 0;\ 3)$, $\boldsymbol{b} = (1;\ 2;\ 1)$ and $\boldsymbol{c} = (0;\ 1;\ 2)$.

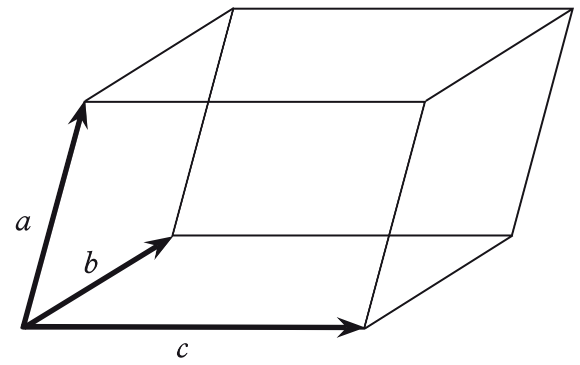

Figure 1.2 Three vectors form a prism

Find the volume of the prism using the definition $Volume = \begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$.

7. For three-dimensional vectors $\bar{u} = (u_1;\ u_2;\ u_3)$ and $\bar{v} = (v_1;\ v_2;\ v_3)$, the cross product vector is defined by:

$$\bar{u} \times \bar{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$ where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit vectors.

a) The cross product $\bar{u} \times \bar{v}$ is perpendicular to the plane formed by $\bar{u}$ and $\bar{v}$. Given $\bar{u} = (1;\ -2;\ 1)$ and $\bar{v} = (3;\ 2;\ 1)$, show that $\bar{u} \times \bar{v} = -4\boldsymbol{i} + 2\boldsymbol{j} + 8\boldsymbol{k}$.

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b) Figure 1.3 shows a force applied at point R to rotate a metal plate about the pivot point O.

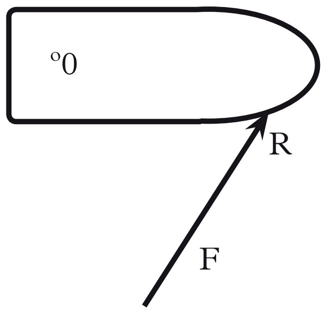

Figure 1.3 A force applied to a metal plate

From the figure, it can be seen that the resulting torque vector is given by $\bar{\tau} = \bar{r} \times \bar{F}$, where:

- $\bar{r}$ is the position vector of the point of application
- $\bar{F}$ is the force vector.

Calculate the torque when $\bar{r} = (-1;\ 2;\ 3)$ and $\bar{F} = (3; -2; -1)$.

Unit 3: Cramer's rule for solving a system

LEARNING OUTCOMES

- Apply Cramer's rule to solve a system of linear equations.

Introduction

You can find solutions to a system of linear equations by solving simultaneous equations or using matrix methods. This unit uses Cramer's rule, which combines determinants to solve for one or more variables.

1. Replace a column in a determinant

The basis of Cramer's rule is that we can form a new determinant by replacing one column.

Worked example 1.6

Consider this 3 × 3 linear system:

$2x + y + z = 3$

$x - z = 0$

$x + 2y = 1$

Solution

Write the right hand constants in a column: $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

Write the coefficients of the equations in a determinant, so that each column represents the coefficients of one variable:

$$\begin{array}{ccc} x & y & z \\ \downarrow & \downarrow & \downarrow \end{array}$$
$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 0 \end{vmatrix}$$

Notice that a zero coefficient represents a missing variable in an equation.

For each variable x, y and z, we can form a new determinant by replacing one coefficient column with the column of constants.

- For x, we replace column 1
- For y, we replace column 2
- For z, we replace column 3:

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 0 \end{vmatrix}; D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix}; D_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

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Evaluate each determinant by expanding along row 1.

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= 3(0 + 2) - 1(0 + 1) + 1(0–0)$$
$$= 3(2) - 1(1) + 0$$
$$= 5$$

$$D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 2(0 + 1) - 3(0 + 1) + 1(1–0)$$
$$= 2(1) - 3(1) + 1(1)$$
$$= 0$$

$$D_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= 2(0) - 1(1–0) + 3(2–0)$$
$$= 5$$

These new determinants are used in Cramer's rule.

ACTIVITY 1.8 GROUP **Investigate Cramer's rule**

A 3 × 3 linear system is given:

$2x + y + z = 3$

$x - y - z = 0$

$x + 2y + z = 0$

1. Show that the values $x = 1$, $y = -2$, $z = 3$ satisfy all three equations.
2. Evaluate the determinant of the coefficients of the system:

   $$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$
3. Write the right hand constants of the system in a column.
4. Form three new determinants D_x, D_y and D_z by replacing each coefficient column in D with the column of constants. (See the Worked example 1.6.)
5. Evaluate the three new determinants by using cofactor expansion along row 1.
6. What do you notice about these three ratios: $\frac{D_x}{D}$, $\frac{D_y}{D}$ and $\frac{D_z}{D}$? This result shows Cramer's rule.

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2. Cramer's rule for finding a variable

Cramer's rule solves for one variable at a time in a system of linear equations.

Definition:

Cramer's rule states that if an $n \times n$ system of linear equations has a non-zero determinant of coefficients D, then $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$, . . . , where the numerator determinants are formed by replacing that variable's coefficients with the column of constants.

Worked example 1.7

1. Consider the 2 × 2 linear system given at the beginning of this module:

$2x - y = 5$

$x + 3y = 6$

Use Cramer's rule to prove that the solution is $x = 3$, $y = 1$.

2. Use Cramer's rule to find the solution to the 3 × 3 linear system:

$2x - 3y + z = -1$

$x - y + z = 1$

$3x - 4z = 0$

Solutions

1. The column of constants is $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$.

The determinant of the coefficients is:

$$D = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = (2)(3) - (1)(-1) = 7$$

$$D_x = \begin{vmatrix} 5 & -1 \\ 6 & 3 \end{vmatrix} = (5)(3) - (6)(-1) = 21$$

$$D_y = \begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix} = (2)(6) - (1)(5) = 7$$

By Cramer's rule, $x = \frac{D_x}{D} = \frac{\begin{vmatrix} 5 & -1 \\ 6 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}} = \frac{21}{7} = 3$

$$y = \frac{D_y}{D} = \frac{\begin{matrix} 2 & 5 \\ 1 & 6 \end{matrix}}{\begin{matrix} 2 & -1 \\ 1 & 3 \end{matrix}} = \frac{7}{7} = 1$$

So, $x = 3$ and $y = 1$.

2. $$D = \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & -4 \end{vmatrix}$$

$$= 2\begin{vmatrix} -1 & 1 \\ 0 & -4 \end{vmatrix} + (-3)(-1)\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix}$$

$$= 2(4–0) + 3(-4–3) + 1(0 + 3)$$

$$= 2(4) + 3(-7) + 3$$

$$= -10$$

The column of constants is $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

Form new determinants by replacing the coefficient column of each variable with the column of constants:

$$D_x = \begin{vmatrix} -1 & -3 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -4 \end{vmatrix} = -1\begin{vmatrix} -1 & 1 \\ 0 & -4 \end{vmatrix} + (-3)(-1)\begin{vmatrix} 1 & 1 \\ 0 & -4 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= -1(4) + 3(-4) + 1(0)$$

$$= -16$$

$$D_y = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 0 & -4 \end{vmatrix} + (-1)(-1)\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= 2(-4) + 1(-4-3) + 1(0-3)$$

$$= -18$$

$$D_z = \begin{vmatrix} 2 & -3 & -1 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} + (-3)(-1)\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix}$$

$$= 2(0) + 3(0-3) - 1(0+3)$$

$$= -12$$

By Cramer's rule:

- $x = \frac{D_x}{D} = \frac{-16}{-10}$
  $= 1{,}6$
- $y = \frac{D_y}{D} = \frac{-18}{-10}$
  $= 1{,}8$
- $z = \frac{D_z}{D} = \frac{-12}{-10}$
  $= 1{,}2$

Check that these values satisfy the three equations in the system.

Note
Always check that the values from Cramer's rule satisfy the equations in the given system.

ACTIVITY 1.9 (SINGLE) Apply Cramer's rule

1. Use Cramer's rule to solve the linear systems:

a) $4x - 3y = 11$
$6x + 5y = 7$

b) $x + 2y + z = 6$
$x - y + z = 0$
$x + y - 2z = 1$

c) $x = 2y + 3z - 1$
$x + y + z = 5$
$2x - z = 3 - y$

Hint: Write the equations in the form $ax + by + cz = k$.

2. An electrical circuit must be set up with currents I_1 and I_2 so that $3I_1 + I_2 = 6$ and $I_1 + 4I_2 = 9$.
Use Cramer's rule to find the required values of I_1 and I_2 correct to three decimal places.

3. Two types of pumps provide water to a reservoir.
   - Four type A and two type B pumps together provide 1 200 ℓ/min.
   - Three type A and five type B pumps can provide 1 600 ℓ/min.

   a) Represent the information as a system of two equations in two variables.

   b) Use Cramer's rule to find the flow rate for each type of pump.

4. Applying Kirchhoff's law to the electrical circuit in Figure 1.4 provides the following system of equations in three currents:
   - $2I_1 + 3I_2 = 6$
   - $2I_1 + 4I_3 = 6$
   - $I_2 + I_3 = I_1$

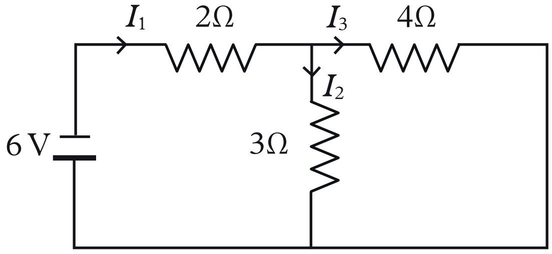

Figure 1.4 An electrical circuit

   a) Represent the system by writing the coefficients in a determinant.
   Hint: First write all equations in the form $aI_1 + bI_2 + cI_3 = k$.

   b) Use Cramer's rule to find the value of the voltage $V = RI_3$ when $R = 26\ \Omega$.

5. One component of an electronic thermometer is a thermistor, which has a resistance given by $R = R_0 + mT$, where R_0 is the resistance at 0 °C.
   - At a temperature of 25 °C the resistance R is 100 Ω.
   - At a temperature of 55 °C the resistance R is 140 Ω.

   Use Cramer's rule to find the values of R_0 and m.

6. A chemist needs to combine a 50% solution of acid with a 25% solution of acid to form 10 ℓ of a 40% acid solution.

   a) If x represents the number of litres of the stronger solution and y represents the number of litres of the weaker solution, complete the system of equations:
   $x + y = \ ...$
   $0{,}5x + \ ...\ y = 4$

   b) Use Cramer's rule to determine how many litres of each solution should be used.

7. An aeroplane travels v km/h in still air. When the wind is w km/h, the aeroplane averages 400 km/h against the wind and 450 km/h with the wind.

   a) Complete the system of equations:
   $v + w = 450$
   $v - w = \ ...$

   b) Use Cramer's rule to calculate the speed of the wind.

8. Mesh analysis of an electrical system yields the following system of equations for currents I_1, I_2 and I_3:
   - $-7 + (I_1 - I_2) + 2(I_1 - I_3) = 0$
   - $(I_2 - I_1) + 2I_2 + 3(I_2 - I_3) = 0$
   - $2(I_3 - I_1) - 6 + 3(I_3 - I_2) + I_3 = 0$

   Use Cramer's rule to solve the system.

Module summary

- For a 2 × 2 system $ax + by = m$ and $cx + dy = n$, the second order determinant that contains the coefficients is defined by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- A linear system has a solution if the determinant that contains the coefficients is non-zero (not zero).
- An entry in row i and column j of a determinant is denoted by a_{ij}.
- The minor M_{ij} of an entry a_{ij} is the value of the smaller determinant that results from deleting row i and column j.
- The cofactor of an entry a_{ij} is defined by $C_{ij} = (-1)^{i+j} \times M_{ij}$.
- A third order (3 × 3) determinant can be found using cofactor expansion along row 1:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$

- Cramer's rule states that if an $n \times n$ system of linear equations has a non-zero coefficient determinant D, then $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ and $z = \frac{D_z}{D}$, where the numerator determinants are formed by replacing that variable's coefficients with the column of constants.

Exam practice questions

1. Find the value of $\begin{vmatrix} \frac{1}{4} & -\frac{1}{2} \\ 6 & 8 \end{vmatrix}$. (2)

2. Given $D = \begin{vmatrix} t & -3 \\ 4 & 1 \end{vmatrix}$ and $E = \begin{vmatrix} 2t & -6 \\ 4 & 1 \end{vmatrix}$, show that $E = 2D$. (2)

3. If the determinant $D = \begin{vmatrix} 2 & -1 & 0 \\ 1 & 3 & 3 \\ 1 & 4 & -3 \end{vmatrix}$ is given, find:

a) M_{13} (the minor of 0) (1)

b) C_{32} (the cofactor of 4) (1)

c) The value of D. (2)

4. Find k if $\begin{vmatrix} k-5 & 6 \\ 3 & k+2 \end{vmatrix} = 0$. (2)

5. Show that $\begin{vmatrix} a & 4a & 7a \\ 2a & 5a & 8a \\ 3a & 6a & 9a \end{vmatrix} = 0$. (3)

6. Use Cramer's rule to find the solution to this system: (5)

- $x + y - z = 6$
- $3x - 2y + z = -5$

- $x + 3y - 2z = 14$

7. Three forces act on a system in the following way:

- $2F_1 + F_2 + F_3 = 1$
- $F_1 = F_2 - 4F_3$
- $F_1 = 3 - 2F_2 + 2F_3$

a) Write the equations in the form $aF_1 + bF_2 + cF_3 = k$ and write the coefficients in a determinant. (1)

b) Use Cramer's rule to find the value of F_3. (3)

8. A triangle in the Cartesian plane with vertices $(x_1;y_1)$, $(x_2;y_2)$ and $(x_3;y_3)$ has an area defined by

$$\text{Area} = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|.$$

Use the formula to calculate the area of the triangle with vertices A(1;−2), B(3;0) and C(−2;4). (3)

Total: 25 marks