Chapter 3: Algebraic processes 1: Quadratic equations

Learning objectives
By the end of this chapter, the students should be able to:
1. Solve quadratic equations by:
   • factorisation
   • using perfect squares
   • completing the square
   • using the quadratic formula.
2. For a quadratic equation given the sum and the product of its roots.
3. Solve word problems by forming and solving suitable quadratic equations.

Teaching and learning materials
Students: Textbook and calculator (if possible)
Teacher: Poster showing the quadratic formula.

Teaching notes and areas of difficulty
• When solving quadratic equations like
  \(x^2 + 4x = 21\) or \(x(6x - 5) = 6\) or \(x(x - 1) = 6\) by factorisation, students do not realise that the right hand side of the equation must always be equal to 0. You must emphasise that we use the zero product principle which states that, if \(A \times B = 0\), then \(A = 0\) or \(B = 0\) or both of them are equal to 0. If students solve the equation \(x(6x - 5) = 6\) and write \(x = 6\) or \(6x - 5 = 6\), it is totally wrong because \(x(6x - 5) = 6 \times 6 = 36 \neq 6\). If students say that \(x = 1\) and \(6x - 5 = 6\), that can also not be a correct method to solve the equation, because there are an infinite number of products which will give 6. For example, \(3 \times 2, 12 \times \frac{1}{2}, 3 \times 4, \frac{1}{4} \times 24\), and so on. If the right-hand side is equal to zero, the possibilities are limited to 0 or 0. Emphasise that they must follow these steps they want to solve \((2x + 3)(x - 1) = 12\):
  
  Step 1 Subtract 12 from both sides of the equation: \((2x + 3)(x - 1) - 12 = 0\)
  Step 2 Remove the brackets by multiplication: \(2x^2 + x - 3 = 12 = 0\)
  Step 3 Add like terms: \(2x^2 + x - 15 = 0\)
  Step 4 Factorise: \((2x - 5)(x + 3) = 0\)
  Step 5 Use the Zero product principle to solve for \(x\):
   
   \(2x - 5 = 0\) or \(x + 3 = 0\)
   
   \(x = \frac{5}{2}\) or \(x = -3\).

• Students use the quadratic formula to solve a quadratic equation when they are asked to solve it by completing the square. Not only do they make the work much more easy for themselves, but only the last two marks for the answer can be given. Emphasise that students read questions thoroughly and do what they are asked to do.

• Students tend to make things difficult for themselves by trying to solve a quadratic equation by completing the square if they cannot factorise. It is very important that students follow these principles when they have to solve quadratic equations:
  1. First try to factorise the equation.
  2. If you cannot factorise the equation (even if it does have factors), you use the quadratic formula to solve it.
  3. You only use the completion of the square to solve a quadratic equation when you are specifically asked to use this method.

• Students tend to make mistakes when using the quadratic formula. Stress that they follow the following steps:
  1. Rewrite the equation in the standard form of \(ax^2 + bx + c = 0\).
  2. Write down the values of \(a, b\) and \(c\) as \(a = \ldots, b = \ldots, c = \ldots\)
  3. Write down the quadratic formula:
     \(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\)
  4. Then, write brackets where the letters were:
     \(x = \frac{-(...) \pm \sqrt{(...)^2 - 4(...)(...)} }{2(...)}\)

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2. For a quadratic equation given the sum and the product of its roots.
3. Solve word problems by forming and solving suitable quadratic equations.
5. Write the values of \( b, b, a, c \) and \( a \) into the brackets.
6. Use a calculator to work out the answer.

Students find word problems that lead to quadratic equations very difficult. You can do the following to help them:

1. Use short sentences when you write a problem for them to solve.
2. Ask them to write down what they do.
3. If an area or perimeter is involved, tell them to make sketches of the problem.
4. Sometimes the problem can also be made clearer by representing it in table form and then completing the table with the information given in the problem. For example:

A woman is 3 times as old as her son. 8 years ago, the product of their ages was 112. Find their present ages:

<table>
<thead>
<tr>
<th>Present age</th>
<th>Age 8 years ago</th>
</tr>
</thead>
<tbody>
<tr>
<td>Woman</td>
<td>3x</td>
</tr>
<tr>
<td>Son</td>
<td>x</td>
</tr>
</tbody>
</table>

Then using the sentence where their ages were compared: \((3x - 8)(x - 8) = 112\).

### Supplementary worked examples

1. Solve these quadratic equations:
   a) \( x(6x - 5) = 0 \)
   b) \( x(6x - 5) = 6 \)
   c) \( (6x + 5)(x + 1) = 15 \)
   d) \( 2x(4x - 11) = -15 \)
   e) \( (x - 3)(x - 2) = 12 \)
   f) \( (x - 8)(x + 1) = -18 \)
   g) \( ax^2 - bx = 0 \)
   h) \( px^2 - px = px \)
   i) \( (x - 3)(x + 2) = x - 3 \)
   j) \( 2(x - 1)(x + 1) = 7(x + 2) - 1 \)

2. If \((x - 3)(y + 4) = 0\), solve for \( x \) and \( y \), if:
   a) \( x = -3 \)
   b) \( x = 3 \)
   c) \( y = -4 \)
   d) \( y = -4 \)

3. Solve for \( x \) by completing the square:
   a) \( 2x^2 + 7x - 4 = 0 \)
   b) \( 2x^2 - 3x - 3 = 0 \) (leave answer in surd form)

### Solutions

1. a) \( x = 0 \) or \( x = \frac{5}{6} \)
   b) \( 6x^2 - 5x - 6 = 0 \)
      \( (2x - 3)(3x + 2) = 0 \)
      \( x = \frac{3}{2} \) or \( x = -\frac{2}{3} \)
   c) \( 6x^2 + 11x + 5 = 15 \)
      \( 6x^2 + 11x - 10 = 0 \)
      \( (2x + 5)(3x - 2) = 0 \)
      \( x = -\frac{5}{2} \) or \( x = \frac{2}{3} \)
   d) \( 8x^2 - 22x + 15 = 0 \)
      \( (2x - 3)(4x - 5) = 0 \)
      \( x = 1\frac{1}{2} \) or \( x = \frac{5}{4} \)
   e) \( x^2 - 5x + 6 = 12 \)
      \( x^2 - 5x - 6 = 0 \)
      \( (x - 6)(x + 1) = 0 \)
      \( x = 6 \) or \( x = -1 \)
   f) \( x^2 - 7x - 8 = -18 \)
      \( x^2 - 7x + 10 = 0 \)
      \( (x - 5)(x - 2) = 0 \)
      \( x = 5 \) or \( x = 2 \)
   g) \( ax(x - 1) = 0 \)
      \( x = 0 \), if \( a \neq 0 \) or \( x \in \mathbb{R} \), if \( a = 0 \) or \( x = 1 \)
   h) \( px^2 - 3px = 0 \)
      \( px(x - 3) = 0 \)
      \( x = 0 \), if \( p \neq 0 \) or \( x \in \mathbb{R} \), if \( p = 0 \) or \( x = 3 \)
   i) \( x^2 - x - 6 = x - 3 \)
      \( x^2 - 2x - 3 = 0 \)
      \( (x - 3)(x + 1) = 0 \)
      \( x = 3 \) or \( x = -1 \)
      \( (x - 3)(x + 2) - (x - 3) = 0 \)
      \( (x - 3)(x + 2 - 1) = 0 \) (take \( x - 3 \) out as a common factor)
      \( (x - 3)(x + 1) = 0 \)
      \( x = 3 \) or \( x = -1 \)
   j) \( 2(x^2 - 1) = 7x + 14 - 1 \)
      \( 2x^2 - 2 - 7x - 13 = 0 \)
      \( 2x^2 - 7x - 15 = 0 \)
      \( (2x + 3)(x - 5) = 0 \)
      \( x = 1\frac{1}{2} \) or \( x = 5 \)

2. a) \( y \in \mathbb{R} \)
   b) \( y = -4 \)
   c) \( x \in \mathbb{R} \)
   d) \( x = 3 \)
3. a) \[ 2x^2 + 7x = 4 \]
\[ x^2 + \frac{7}{2}x = 2 \]
\[ x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = 2 + \left(\frac{7}{4}\right)^2 \]
\[ \left(x + \frac{7}{4}\right)^2 = 2 + \frac{49}{16} = \frac{81}{16} \]
\[ x + \frac{7}{4} = \pm \sqrt{\frac{81}{16}} \]
\[ x = -\frac{7}{4} \pm \frac{9}{4} \]
\[ x = -4 \text{ or } x = \frac{1}{2} \]

b) \[ 2x^2 - 3x = 3 \]
\[ x^2 - \frac{3}{2}x = \frac{3}{2} \]
\[ x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = \frac{3}{2} + \frac{9}{16} \]
\[ \left(x - \frac{3}{4}\right)^2 = \frac{33}{16} \]
\[ x - \frac{3}{4} = \pm \sqrt{\frac{33}{16}} \]
\[ x = \frac{3}{4} \pm \frac{\sqrt{33}}{4} \]
\[ x = \frac{3 \pm \sqrt{33}}{4} \]