Chapter 7: Algebraic processes 2: Simultaneous linear and quadratic equations

Learning objectives
By the end of this chapter, the students should be able to:
1. Solve simultaneous linear equations using elimination, substitution and graphical methods.
2. Solve simultaneous linear and quadratic equations using substitution and graphical methods.
3. Solve word problems leading to simultaneous linear equations and simultaneous linear and quadratic equations.

Teaching and learning materials
Students: Textbook and graph paper.
Teacher: Graph chalkboard or transparencies of graph paper and transparency pens, if an overhead projector is available. Wire that can bend and hold its form: to draw the curves of quadratic functions or plastic parabolic curves, if you have them.

Areas of difficulty
• When solving simultaneous linear equations, students experience these difficulties:
  1. They do not number the original equation, or the equations that they create, and the result is that they make unnecessary mistakes. Insist repeatedly that students number equations.
  2. They do not know whether to add or subtract equations to eliminate one of the variables. Teach them that:
     ▪ If in the two equations the coefficients of the same variable are equal and also have the same sign, then they subtract the two equations from each other. For example:
       \[ 2x - 3y = 10 \ldots \text{1} \]
       \[ 2x + 5y = -14 \ldots \text{2} \]
       \[ \text{1} - \text{2}: -8y = 24 \]
       \[ \therefore y = -3 \]
     Or
       \[ -2x - 3y = 10 \ldots \text{1} \]
       \[ -2x + 5y = -14 \ldots \text{2} \]
       \[ \text{1} - \text{2}: -8y = 24 \]
       \[ \therefore y = -3 \]

  3. When subtracting two equations, students sometimes forget that the signs of the equation at the bottom change. Again, emphasise that when subtracting from above, the left-hand side can also be written as \[ 2x - 3y - (2x + 5y) = 2x - 3y - 2x - 5y = -8y. \]
The right-hand side can be written as \[ 10 - (-14) = 10 + 14 = 24. \]
  4. In equations such as \[ 3x - 2y = 24 \]
     and \[ 4x - 9y = 36, \]
     students must realise that to eliminate \(x\), they must first get the LCM of 3 and 4, which is 12. Then the first equation must be multiplied by 4, because \[4 \times 3 = 12\]
     and the second equation must be multiplied by 3, because \[3 \times 4 = 12:\]
     \[ 3x - 2y = 24 \ldots \text{1} \]
     \[ 4x - 9y = 36 \ldots \text{2} \]
     \[ \text{1} \times 4: 12x - 8y = 96 \ldots \text{3} \]
     \[ \text{2} \times 3: 12x - 27y = 108 \ldots \text{4} \]
     They could of course have used the LCM of 2 and 9, which is 18 and multiplied the first equation by 9 and the second by 2 to eliminate \(y\).

• Students make mistakes if they substitute the value of the variable they found into an equation to find the value of the other variable.

If the coefficients of the same variable are equal but have opposite signs, they must add the two equations. For example:

\[ -2x - 3y = 10 \ldots \text{1} \]
\[ 2x + 5y = -14 \ldots \text{2} \]
\[ \text{1} + \text{2}: 2y = -4 \]
\[ \therefore y = -2 \]
The main reason for these errors is the fact that they do not use brackets for substitution. Let us say that they found that \( x \), for example, is equal to \(-3\). They have to substitute the value of \( x \) into \(-3x + 2y = -4\) to find the value of \( y \).

Then, to prevent errors, they should substitute the value of \( x \) like this:
\[-3(-3) + 2y = -4.\]

* When writing a two-digit number, students should realise the following:
  - If the tens digit is \( x \) and the units digit is \( y \), they cannot write the number as \( xy \).
  - In algebraic language \( xy \), means \( x \times y \).
  - Take an example such as 36 and explain that in our number system 36 means \( 3 \times 10 + 6 \).
  - So a two digit number must be written as \( 10x + y \).
* When solving simultaneous linear and quadratic equations, students tend to make matters very difficult for themselves by choosing to make the variable (which will result in them working with a fraction) the subject of the linear equation. For example:
  - If \( 3x + y = 10 \) then solve for \( y \) and not \( x \).
  - If we solve for \( y \), then \( y = 10 - 3x \).
  - On the other hand, if we solve for \( x \), then \( x = \frac{10 - y}{3} \).
  - So, if the value of \( x \) has to substituted into a quadratic equation, the work will be much more difficult.
  - So, insist that students always ask themselves what the easiest option is, and then make that variable the subject of the formula.
* Sometimes students find it difficult to factorise a quadratic equation. Teach them to use the quadratic formula if they find it difficult to get the factors.
* Students find it difficult to draw a perfect curve through the points on a parabola. Teach them to be very careful when drawing this curve.
  - It would be ideal if you have a couple of plastic curves in your class.
  - If not, you could always keep soft wire that can then be bent to fit all the points on the curve. Then a student could use a pencil and trace the curve made by the wire.
* If the graph of the quadratic function must be drawn to solve simultaneous equations (one linear and the other quadratic), the range of the \( x \)-values should be given so that the turning point of the graph or the quadratic function is included.
* Students almost always find word problems difficult. You could help them by using short, single sentences when setting test questions. You could also help them by leading them to find the equations by asking sub questions in your test or examination question.
  - In class, you must teach them how to recognise the variables. Then help them to translate the words sentence by sentence into Mathematics using these variables.
  - For problems where the area or perimeter of a plane figures is involved, insist that they make drawings to represent the situation.
  - The facts of some problems can also be represented in a table that makes it easier to understand. Number 14 of Exercise 7c can be represented in table form like this:

<table>
<thead>
<tr>
<th>Distance</th>
<th>Time</th>
<th>Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run</td>
<td>8 km</td>
<td>( \frac{8}{5} ) x</td>
</tr>
<tr>
<td>Walk</td>
<td>2 km</td>
<td>( \frac{2}{7} ) y</td>
</tr>
<tr>
<td>Run</td>
<td>4 km</td>
<td>( \frac{4}{2} ) x</td>
</tr>
<tr>
<td>Walk</td>
<td>6 km</td>
<td>( \frac{6}{7} ) y</td>
</tr>
</tbody>
</table>

\[
\frac{8}{5}x + \frac{2}{7}y = \frac{5}{6} \\
\frac{4}{5}x + \frac{6}{7}y = \frac{3}{4}
\]

When solving a speed–time–distance problem, make sure that the speed is measured in the same unit as time. In the problem above, the times were in hours and minutes. In the equations, both times are in hours.

If your class struggles too much with word problems, do not spend unnecessary time on this work. Rather spend more time on the basics of this chapter where students can more easily earn marks to pass Mathematics.
Supplementary worked examples

1. Solve for \(x\) and \(y\) in these simultaneous equations:
   a) \(\frac{81x - 3y}{9} = 27\) and \(\frac{64x - 2y}{16} = 1\)
   b) \(3x + 2y = 1\) and \(2x + 5 = \left(\frac{1}{8}\right)^y\)
   c) \((2x + 3y)(x - 2y) = 9\) and \(x - 2y = 3\)

2. Solve for \(x\) and \(y\): i) algebraically and ii) graphically.
   a) \(2x - 3y = 6\) and \(4x - 6y = 12\)
   b) \(3x - 2y = -4\) and \(6x - 4y = 8\)

3. Use this graph to solve the following equations:
   a) \(y = x^2 + 2x - 8\) and \(y = -8\)
   b) \(y = x^2 + 2x - 8\) and \(y = -9\)
   c) \(y = x^2 + 2x - 8 = 0\)
   d) \(x^2 + 2x - 12 = 0\)
   e) \(x^2 + 2x - 8 = -2x - 3\)

Solutions

1. a) \((3^4)^{2x - 3y} = 3^5\)
   \(3^{8x - 12y} = 3^5\)
   \(8x - 12y = 5\) \(\ldots\) 1
   \((2^6)^{x - 2y} = 2^4\)
   \(2^{6x - 12y} = 2^4\)
   \(6x - 12y = 4\) \(\ldots\) 2
   1 - 2: \(2x = 1\)
   \(\therefore x = \frac{1}{2}\)
   Substitute \(x = \frac{1}{2}\) into 2:

   \[6\left(\frac{1}{2}\right) - 12y = 4\]
   \[-12y = 4 - 3\]
   \(\therefore y = -\frac{1}{12}\)

b) \(3^x + 2y = 3^0\)
   \(x + 2y = 0\) \(\ldots\) 1
   \(2^x + 5 = (2^{-3})^y\)
   \(2^x + 5 = 2^{-3y}\)
   \(-x + 5 = -3y\)
   \(-x + 3y = -5\) \(\ldots\) 2
   1 + 2: \(5y = -5\)
   \(\therefore y = -1\)

Substitute \(y = -1\) into 1:
\(x = 2\)

2. a) i) \(2x - 3y = 6\) \(\ldots\) 1
   \(4x - 6y = 12\) \(\ldots\) 2
   2 + 2: \(2x - 3y = 6\)
   Answer: \(x \in \mathbb{R}\) and \(y \in \mathbb{R}\).

ii) \(-3y = -2x + 6 \Rightarrow y = \frac{2}{3}x - 2\)

\[
\begin{array}{ccc}
\hline
x & y = \frac{2}{3}x - 2 \\
-3 & -4 \\
0 & -2 \\
3 & 0 \\
\hline
\end{array}
\]
The two equations are actually just multiples of each other. So, they are represented by the same graph. So, all the real values of \( x \) and all the real values of \( y \) will satisfy both equations simultaneously.

b) i) \[ 3x - 2y = -4 \quad \text{1} \]
\[ 6x - 4y = 8 \quad \text{2} \]
\[ \text{1} \times 2: 6x - 4y = -3 \quad \text{3} \]

However, \( 6x - 4y \) cannot be equal to \(-8\) and \(8\). So, there is no solution.

ii) \[ -2y = -3x - 4 \]
\[ y = \frac{3}{2}x + 2 \]
\[ -4y = -6x + 8 \]
\[ \therefore y = \frac{3}{2}x - 2 \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \frac{3}{2}x + 2 )</td>
<td>-1</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>0</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \frac{3}{2}x - 2 )</td>
<td>-8</td>
<td>-2</td>
<td>4</td>
</tr>
</tbody>
</table>

In the graph above, you can see that the two graphs will never intersect because they are parallel. So, there is no solution, or no point that lies on both graphs, or no \( x \)- and \( y \)-values that satisfy the two equations simultaneously.

3. a) Draw \( y = -8 \):

Read from graph: Therefore, \( x = -2 \) or \( x = 0 \).

b) Draw \( y = -9 \).

Read from graph: Therefore, \( x = -1 \).

c) \( y = 0 \) is the \( x \)-axis: Therefore, \( x = -4 \) or \( x = 2 \).

d) \[ x^2 + 2x - 12 + 4 = 4 \]
\[ x^2 + 2x - 8 = 4 \]

Draw \( y = 4 \). Therefore, \( x = -4.6 \) or \( x = 2.6 \).

e) Draw the graph of \( y = -2x - 3 \):

Therefore, \( x = -5 \) or \( x = 1 \) and \( y = 7 \) or \( y = -5 \).