New General Mathematics

FOR SENIOR SECONDARY SCHOOLS

2

TEACHER'S GUIDE

ALWAYS LEARNING

PEARSON

New General Mathematics

for Secondary Senior Schools 2

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Review of SB1 and SB2

1. Learning objectives

1. Number and numeration
2. Algebraic processes
3. Geometry and mensuration
4. Statistics and probability

2. Teaching and learning materials

Teachers should have the Mathematics textbook of the Junior Secondary School Course and Book 1 and Book 2 of the Senior Secondary School Course.

Students should have:

1. Book 2
2. An Exercise book
3. Graph paper
4. A scientific calculator, if possible.

3. Glossary of terms

Algebraic expression A mathematical phrase that contains ordinary numbers, variables (such as x or y) and operators (such as add, subtract, multiply, and divide). For example, $3x^2y - 3y^2 + 4$.

Angle A measure of rotation or turning and we use a protractor to measure the size of an angle.

Angle of elevation The angle through which the eyes must look upward from the horizontal to see a point above.

Angle of depression The angle through which the eyes must look downward from the horizontal to see a point below.

Balance method The method by which we add, subtract, multiply or divide by the same number on both sides of the equation to keep the two sides of the equation equal to each other or to keep the two sides balanced. We use this method to make the two sides of the equation simpler and simpler until we can easily see the solution of the equation.

Cartesian plane A **coordinate** system that specifies each point in a plane uniquely by a pair of numerical coordinates, which are the perpendicular distances of the point from two fixed perpendicular directed lines or axes, measured in the same unit of length. The word Cartesian comes from the inventor of this plane namely René Descartes, a French mathematician.

Coefficient a numerical or constant or quantity $\neq 0$ placed before and multiplying the variable in an algebraic expression (for example, 4 in $4x^y$).

Common fraction (also called a vulgar fraction or simple fraction) Any number written as $\frac{a}{b}$ where a and b are both whole numbers and where $a < b$.

Coordinates of point A, for example, (1, 2) give its position on a Cartesian plane. The first coordinate (x-coordinate) always gives the distance along the x-axis and the second coordinate (y-coordinate) gives the distance along the y-axis.

Data Distinct pieces of information that can exist in a variety of forms, such as numbers. Strictly speaking, data is the plural of *datum*, a single piece of information. In practice, however, people use *data* as both the singular and plural form of the word.

Decimal place values A positional system of notation in which the position of a number with respect to the decimal point determines its value. In the decimal (base 10) system, the value of each digit is based on the number 10. Each position in a decimal number has a value that is a power of 10.

Denominator The part of the fraction that is written below the line. The 4 in $\frac{3}{4}$, for example, is the denominator of the fraction. It also tells you what kind of fraction it is. In this case, the kind of fraction is quarters.

Direct proportion The relationship between quantities whose ratio remains constant. If a and b are directly proportional, then $\frac{a}{b}$ = a constant value (for example, k).

Direct variation Two quantities, a and b vary directly if, when a changes, then b changes in the same ratio. That means that:

- If a doubles in value, b will also double in value.
- If a increases by a factor of 3, then b will also increase by a factor of 3.

Directed numbers Positive and negative numbers are called directed numbers and could be shown on a number line. These numbers have a certain direction with respect to zero.

- If a number is positive, it is on the right-hand side of 0 on the number line.

- If a number is negative, it is on the left-hand side of the 0 on the number line.

Edge A line segment that joins two vertices of a solid.

Elimination the process of solving a system of simultaneous equations by using various techniques to successively remove the variables.

Equivalent fractions Fractions that are multiples of each other, for example, $\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{3 \times 3}{4 \times 3} \ldots =$ and so on.

Expansion of an algebraic expression means that brackets are removed by multiplication

Faces of a solid A flat (planar) surface that forms part of the boundary of the **solid** object; a three-dimensional solid bounded exclusively by flat faces is a polyhedron.

Factorisation of an algebraic expression means that we write an algebraic expression as the product of its factors.

Graphical method used to solve simultaneous linear equations means that the graphs of the equations are drawn. The solution is where the two graphs intersect (cut) each other.

Highest Common Factor (HCF) of a set of numbers is the highest factor that all those numbers have in common or the highest number that can *divide into* all the numbers in the set. The HCF of 18, 24 and 30, for example, is 6.

Inverse proportion The relationship between two variables in which their product is a constant. When one variable increases, the other decreases in proportion so that the product is unchanged. If b is inversely proportional to a, the equation is in the form $b = \frac{k}{a}$ (where k is a constant).

Inverse variation Two quantities a and b vary inversely if, when a changes, then b changes by the same ratio inversely. That means that:

- If a doubles, then b halves in value.
- If a increases by a factor of 3, then b decreases by a factor of $\frac{1}{3}$.

Joint variation of three quantities x, y and z means that x and y are directly proportional, for example, and x and z are inversely proportional, for example. So $x \propto \frac{y}{z}$ or $x = k\frac{y}{z}$, where k is a constant.

Like terms contain identical letter symbols with the same exponents. For example, $-3x^2y^3$ and $5x^2y^3$ are like terms but $3x^2y^3$ and $3xy$ are not like terms. They are **unlike** terms.

Lowest Common Multiple (LCM) of a set of numbers is the smallest multiple that a set of numbers have in common or the smallest number *into* which all the numbers of the set can *divide* without leaving a remainder. The LCM of 18, 24 and 30, for example, is 360.

Median The median is a measure of central tendency. To find the median, we arrange the data from the smallest to largest value.

- If there is an odd number of data, the median is the middle value.
- If there is an even number of data, the median is the average of the two middle data points.

Mode The value (data point) that occurs the most in a set of values (data) or is the data point with the largest frequency.

Multiple The multiple of a certain number is that number *multiplied by* any other whole number. Multiples of 3, for example, are 6, 9, 12, 15, and so on.

Net A plane shape that can be folded to make the solid.

Numerator The part of the fraction that is written above the line. The 3 in $\frac{3}{8}$, for example, is the numerator of the fraction. It also tells how many of that kind of fraction you have. In this case, you have 3 of them (eighths).

Orthogonal projection A system of making engineering drawings showing several different views (for example, its plan and elevations) of an object at right angles to each other on a single drawing.

Parallel projection Lines that are parallel in reality are also parallel on the drawing

Pictogram (or pictograph) Represents the frequency of data as pictures or symbols. Each picture or symbol may represent one or more units of the data.

Pie chart A circular chart divided into sectors, where each sector shows the relative size of each value. In a pie chart, the angle of the each sector is in the same ratio as the quantity the sector represents.

Place value Numbers are represented by an ordered sequence of digits where both the digit and its place value have to be known to determine its value. The 3 in 36, for example, indicates 3 tens and 6 is the number of units.

Terms in an algebraic expression are numbers and variables which are separated by + or − signs.

Satisfy an equation, means that there is a certain value(s) that will make the equation true. In the equation $4x + 3 = -9$, $x = -3$ satisfies the equation because $4(-3) + 3 = -9$.

Simplify means that you are writing an algebraic expression in a form that is easier to use if you want to do something else with the expression. If you want to add fractions, for example, you need to write all the fractions with the same denominator to be able to add them. Then the simplest form of $\frac{3}{4}$ is $\frac{9}{12}$, if 12 is the common denominator.

Simultaneous linear equations are equations that you solve by finding the solution that will make them simultaneously true. In $2x - 5y = 16$ and $x + 4y = -5$, $x = 3$ and $y = -2$ satisfy both equations simultaneously.

SI units The international system of units of expressing the magnitudes or quantities of important natural phenomena such as length in metres, mass in kilograms and so on.

Solve an equation means that we find the value of the unknown (variable) in the equation that will make the statement true. In the equation $3x - 4 = 11$, the value of the unknown (in this case, x) that will make the statement true, is 5, because $3(5) - 4 = 11$.

Variable In algebra, variables are represented by letter symbols and are called variables because the values represented by the letter symbols may vary or change and therefore are not constant.

Vertex (plural vertices) A point where two or more edges meet.

***x*-axis** The horizontal axis on a Cartesian plane.

***y*-axis** The vertical axis on a Cartesian plane.

Teaching notes

You should be aware of what your class knows about the work from previous years. It would be good if you could analyse their answer papers from the previous end of year examination to determine where the class lacks the necessary knowledge and ability in previous work. You can then analyse the students' answers to determine where they experience difficulties with the work, and then use this chapter to concentrate on those areas.

A good idea would be that you review previous work by means of the summary given in each section. Then you let the students do Review test 1 of that section and you discuss the answers when they finished it. You then let the students write Review test 2 as a test, and you let them mark it under your supervision.

Chapter 1 Numerical processes 1: Logarithms

Learning objectives

By the end of this chapter, the students should be able to:

1. Recall the use of logarithm tables to perform calculations with numbers greater than 1.
2. Compare characteristics of logarithms with corresponding numbers in standard form.
3. Use logarithm tables to perform calculations with numbers less than 1, including:
   - multiplication and division
   - powers and roots of numbers.
4. Solve simple logarithmic equations.

Teaching and learning materials

Students: Copy of textbook, exercise book and writing materials.
Teacher: Copy of textbook and a transparency showing logs and antilogs of numbers.

Areas of difficulty

- Students tend to forget what the word logarithm really means. Emphasise the following: If $10^{2.301} = 200$, then $\log_{10} 200 = 2.301$. In words: log base 10 of 200 is the exponent to which 10 must be raised to give 200.
- Students tend to forget what antilog means. If, for example, $10^{2.301} = 200$, the antilog means that we want to know what the answer of $10^{2.301}$ is.
- Students tend to forget why they add logarithms of numbers, if they multiply the numbers; and why they subtract logarithms of numbers, if they divide these numbers by each other. Emphasise that logarithms are exponents and that the first two exponential laws are:
  - Law 1: $a^x \times a^y = a^{x+y}$. For example, $a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a)$ $= a \times a \times a \times a \times a \times a \times a = a^7$.
  - Law 2: $a^x \div a^y = a^{x-y}$, where $x > y$. For example, $\frac{a^6}{a^2} = \frac{a \times a \times a \times a \times a \times a}{a \times a} = a^4 = a^{6-2}$.
- As logarithms to the base 10 are the exponents of 10:
  - We add the logs of the numbers, if we multiply the numbers.
  - We subtract the logs of the numbers, if we divide the numbers by each other.
- When working out a number to a certain power, students tend to forget why they multiply the log of the number with the power. Again, emphasise this exponential law (since logs to the base 10 are the same as the exponents of 10 that will give the number): $(a^m)^n = a^{mn}$. For example, $(a^4)^2 = (a \times a \times a \times a) \times (a \times a \times a \times a) =$ $a \times a \times a \times a \times a \times a \times a \times a = a^{4 \times 2} = a^8$ and $\sqrt{a^6} = (a^6)^{\frac{1}{2}} = a^{6 \times \frac{1}{2}} = a^3$.
- When adding, subtracting, multiplying and dividing logarithms with negative characteristics, students may experience some difficulty.
  - Emphasise that the characteristic of the logarithm must be added, subtracted, multiplied and divided separately from the mantissa and treated as directed numbers.
  - Emphasise and explain examples, such as $\bar{2}.2 - \bar{4}.5$ very carefully: $(\bar{3} + 1.2) - (\bar{4} + 0.5)$ (you add $\bar{1}$ to $\bar{2}$ and $+1$ to 0.2 – then you add 0 and do not change anything) = $(\bar{3} - (-4)) + (1.2 - 0.5) = (-3 + 4) + (0.7) = 1.7$.
- In the beginning, students may find it difficult to write down their calculations with logs in table form. Give them a lot of guidance and emphasise that it is essential that they write all their calculations out in table form to prevent mistakes.
- Students tend to become confused if they have to write the exponential form $N = a^x$ in its logarithmic form, $\log_a N = x$, especially if the base is not 10 anymore. Give them enough examples to practise this.

Supplementary worked examples

1. Write these in logarithmic form:
   a) $2^5 = 32$ b) $5^0 = 1$
   c) $3^{-3} = \frac{1}{27}$ d) $25^{-\frac{1}{2}} = \frac{1}{5}$
2. Write these logs in exponential form:
   a) $\log_{\frac{1}{3}} 81 = -4$ b) $\log_a a^x = x$
   c) $\log_{\frac{1}{2}} 8 = -3$ d) $\log 10 = 1$
3. Write down the values of:
   a) $\log_2 1$ b) $\log_{\frac{1}{5}} 625$
   c) $\log_3 9 + \log_4 64 - \log_3 3$
   d) $\log_5 25 + \log_7 49 - \log_{12} 144$

Solutions

1. a) $\log_2 32 = 5$ b) $\log_5 1 = 0$
   c) $\log_3 \frac{1}{27} = -3$ d) $\log_{25} \frac{1}{5} = -\frac{1}{2}$
2. a) $\left(\frac{1}{3}\right)^{-4} = 81$ b) $a^x = a^x$
   c) $\left(\frac{1}{2}\right)^{-3} = 8$ d) $10^1 = 10$
3. a) $2^0 = 1$, Answer = 0
   b) $\left(\frac{1}{5}\right)^{-4} = 625$, Answer = −4
   c) $2 + 3 - 1 = 4$
   d) $2 + 2 - 2 = 2$

Chapter 2 Circle geometry 1: Chords, arcs and angles

Learning objectives

By the end of this chapter, the students should be able to:

1. Define and name the lines and regions of a circle, including cyclic quadrilaterals.
2. Recall, use and apply mid-point properties of chords.
3. Recall, use and apply the following properties of angles subtended by arcs and chords at the centre and circumference of a circle:
   - angle at centre = 2 × angle at circumference
   - angles in the same segment are equal
   - angle in a semi-circle = 90°.
4. Recall, use and apply the following properties of cyclic quadrilaterals:
   - opposite angles are supplementary
   - exterior angle = opposite interior angle.

Teaching and learning materials

Students: Mathematical set (including a protractor and compass).
Teacher: Posters, cardboard models, chalkboard instruments (especially a protractor and compass), computer instructional materials where available.

Areas of difficulty

- Students tend to forget theorems and deductions made from them.

  Students must always give a reason for each statement that they make that is based on some or other theorem, and they either do not give the reason or do not know how.

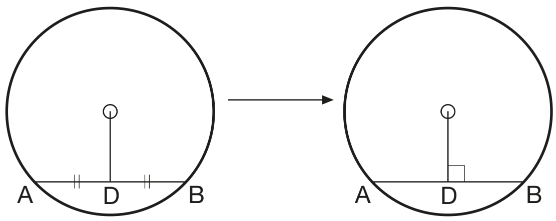
  To help the students, you could give them summaries of the theorems and suggestions of the reasons they can give. Below are suggestions:

1. Theorem 10 and the rules that follow can be remembered as follows:

   Three facts:
   - Centre circle
   - Midpoint chord
   - Perpendicular chord

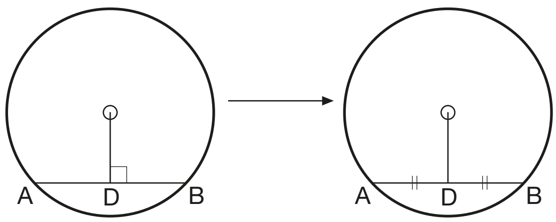
   If **two** of these facts are **present**, then the **other one** is also **present**.

a) Given: Centre circle, Centre chord: ⊥ chord



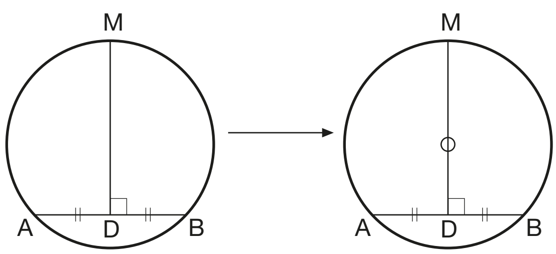
Reason given if this fact is used: **OD bisects AB**

b) Given: Centre circle, ⊥ chord Centre chord



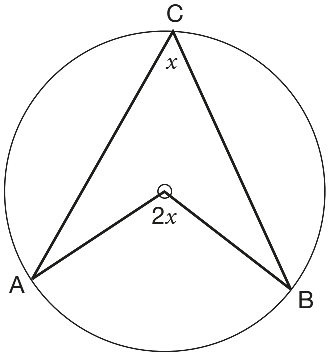
Reason given if this fact is used: **OD ⊥ AB**

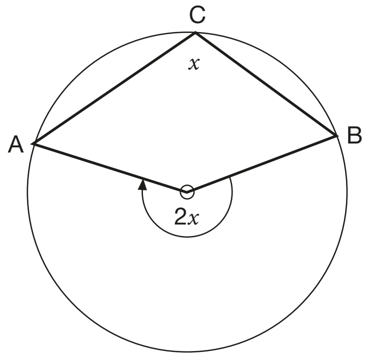
c) Given: Centre chord, ⊥ chord centre circle

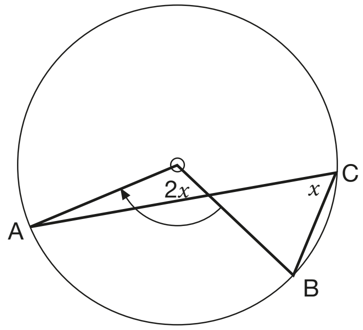


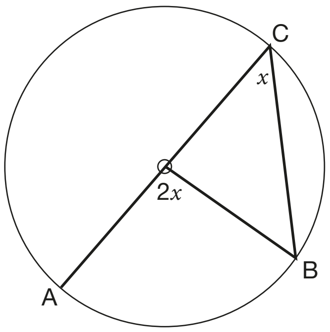
Reason given if this fact is used: **AD = DB, MD ⊥ AB**

2. Angle at the centre of a circle subtended by a certain arc is equal to twice the angle at the circumference subtended by the same arc.



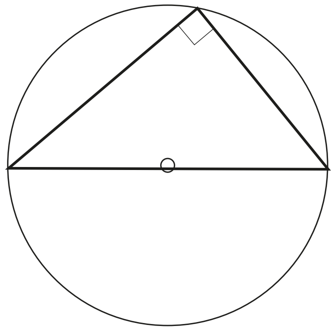






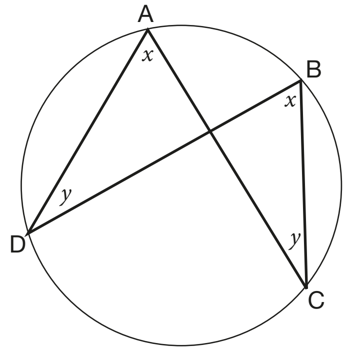
Reason given if this theorem is used:
∠ **at centre** = **2** × ∠ **at circumf. on AB**
(it is always easier to write reasons as short as possible)

3. Deductions from this theorem:
   a)



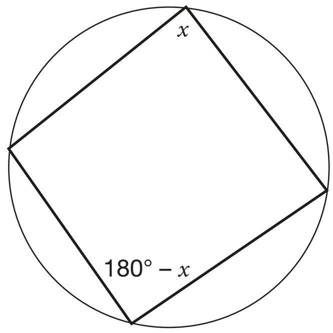
Reason given if this fact is used: ∠ **in semi - circle**

   b)



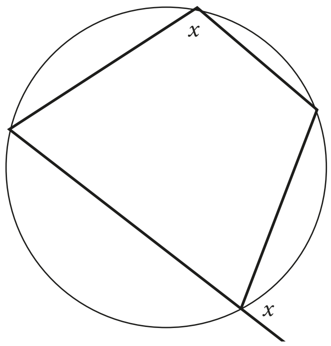
Reason given if this fact is used:
∠**'s on DC** = or ∠**'s on AB** =

4. Cyclic quadrilateral theorems:
   a)



Reason given if this theorem is used:
opp. ∠**'s cyclic quad suppl**

   b)



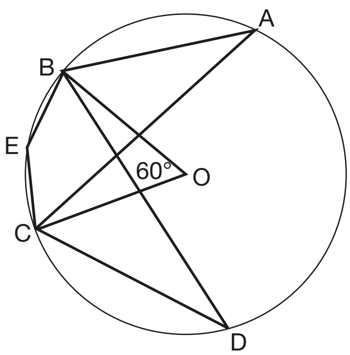
Reason given if this theorem is used:
Ext ∠ **cyclic quad** = **opp interior** ∠

- Students find it difficult to recognise the angle(s) at the circumference subtended by the same arc as the angle at the centre of the circle. You can help them by teaching them to use their thumb and forefinger to trace the angle at the centre starting from the two points of the arc that subtend this angle. Then they start at the same two points and trace the angle at the circumference of the circle subtended by the same arc (chord).
- Students find it difficult to recognise the angle(s) at the circumference subtended by the same arc. You can help them by teaching them to use their thumb and forefinger to trace the one angle starting from the two points of the arc that subtends this angle. Then they start at the same two points and trace the other angle(s) at the circumference subtended by the same arc (chord).
- Students find it difficult to see the exterior angle of the cyclic quadrilateral. You can help them to recognise any exterior angle by letting them draw any quadrilateral. Then they must lengthen its sides to get as many as possible of its exterior angles.
- Students find it difficult to see which interior angle is the angle opposite the exterior angle of the quadrilateral. Teach them that:
    - It is not any of the angles which share a side or a produced side with the exterior angle.
    - Neither is it the angle adjacent to the exterior angle.
    - So it could only be one other angle.

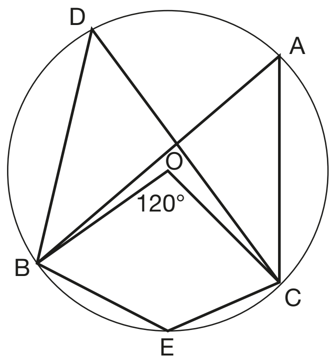
Supplementary worked examples

If students find it difficult in the beginning to recognise angles at the centre of a circle and the angles at the circumference of the circle that are equal to twice the angle at the centre of the circle, you can let them do the following problems.

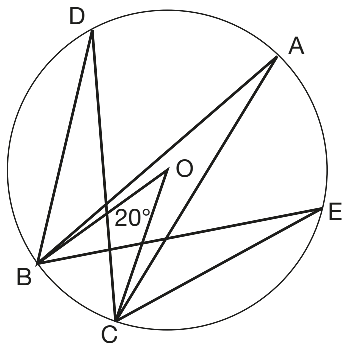
1. If ∠O = 60°, find the sizes of, ∠A, ∠D reflex ∠O and ∠E.



2. If ∠O = 120°, find the sizes of ∠A, ∠D, reflex ∠O and ∠E.



3. If ∠O = 20°, find the sizes of ∠D, ∠A and ∠E.



Solutions

1. ∠A = 30° = ∠ D (∠ at centre = 2 × ∠ at circumference on BC)
   Reflex ∠O = 300° (∠'s around a point = 360°)
   ∠E = 150° (∠ at centre = 2 × ∠ at circumference on AD)
2. ∠D = ∠A = 60° (∠ at centre = 2 × ∠ at circumference on BC)
   Reflex ∠O = 240° (∠'s around a point = 360°)
   ∠E = 120° (∠ at centre = 2 × ∠ at circumference on AD)
3. ∠D = ∠A = ∠E = 10° (∠ at centre = 2 × ∠ at circumference on BC

Chapter 3 Algebraic processes 1: Quadratic equations

Learning objectives

By the end of this chapter, the students should be able to:

1. Solve quadratic equations by:
   - factorisation
   - using perfect squares
   - completing the square
   - using the quadratic formula.
2. For a quadratic equation given the sum and the product of its roots.
3. Solve word problems by forming and solving suitable quadratic equations.

Teaching and learning materials

Students: Textbook and calculator (if possible)
Teacher: Poster showing the quadratic formula.

Teaching notes and areas of difficulty

- When solving quadratic equations like $x^2 + 4x = 21$ or $x(6x - 5) = 6$ or $x(x - 1) = 6$ by factorisation, students do not realise that the right hand side of the equation must always be equal to 0. You must emphasise that we use the zero product principle which states that, if $A \times B = 0$, then $A = 0$ or $B = 0$ or both of them are equal to 0. If students solve the equation $x(6x - 5) = 6$ and write $x = 6$ or $6x - 5 = 6$, it is totally wrong because $x(6x - 5) = 6 \times 6 = 36 \neq 6$. If students say that $x = 1$ and $6x - 5 = 6$, that can also not be a correct method to solve the equation, because there are an infinite number of products which will give 6. For example, 3×2, $12 \times \frac{1}{2}$, 3×4, $\frac{1}{4} \times 24$, and so on. If the right-hand side is equal to zero, the possibilities are limited to 0 or 0. Emphasise that they must follow these steps they want to solve $(2x + 3)(x - 1) = 12$:

  Step 1 Subtract 12 from both sides of the equation: $(2x + 3)(x - 1) - 12 = 0$
  Step 2 Remove the brackets by multiplication: $2x^2 + x - 3 - 12 = 0$
  Step 3 Add like terms: $2x^2 + x - 15 = 0$
  Step 4 Factorise $(2x - 5)(x + 3) = 0$
  Step 5 Use the Zero product principle to solve for x:
  $2x - 5 = 0$ or $x + 3 = 0$
  $x = 2\frac{1}{2}$ or $x = -3$.

- Students use the quadratic formula to solve a quadratic equation when they are asked to solve it by completing the square. Not only do they make the work much more easy for themselves, but only the last two marks for the answer can be given. Emphasise that students read questions thoroughly and do what they are asked to do.
- Students tend to make things difficult for themselves by trying to solve a quadratic equation by completing the square if they cannot factorise. It is very important that students follow these principles when they have to solve quadratic equations:
  1. First try to factorise the equation.
  2. If you cannot factorise the equation (even if it does have factors), you use the quadratic formula to solve it.
  3. You only use the completion of the square to solve a quadratic equation when you are specifically asked to use this method.
- Students tend to make mistakes when using the quadratic formula. Stress that they follow the following steps:
  1. Rewrite the equation in the standard form of $ax^2 + bx + c = 0$.
  2. Write down the values of a, b and c as $a = \ldots$, $b = \ldots.$, $c = \ldots.$
  3. Write down the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  4. Then, write brackets where the letters were: $x = \frac{-(\ldots) \pm \sqrt{(\ldots)^2 - 4(\ldots)(\ldots)}}{2(\ldots)}$

5. Write the values of b, b, a, c and a into the brackets.
6. Use a calculator to work out the answer.

Students find word problems that lead to quadratic equations very difficult. You can do the following to help them:

1. Use short sentences when you write a problem for them to solve.
2. Ask them to write down what they do.
3. If an area or perimeter is involved, tell them to make sketches of the problem.
4. Sometimes the problem can also be made clearer by representing it in table form and then completing the table with the information given in the problem. For example:
   A woman is 3 times as old as her son. 8 years ago, the product of their ages was 112. Find their present ages:

	Present age	Age 8 years ago
Woman	$3x$	$3x - 8$
Son	x	$x - 8$

Then using the sentence where their ages were compared: $(3x - 8)(x - 8) = 112$.

Supplementary worked examples

1. Solve these quadratic equations:
   a) $x(6x - 5) = 0$
   b) $x(6x - 5) = 6$
   c) $(6x + 5)(x + 1) = 15$
   d) $2x(4x - 11) = -15$
   e) $(x - 3)(x - 2) = 12$
   f) $(x - 8)(x + 1) = -18$
   g) $ax^2 - ax = 0$
   h) $px^2 - px = 2px$
   i) $(x - 3)(x + 2) = x - 3$
   j) $2(x - 1)(x + 1) = 7(x + 2) - 1$
2. If $(x - 3)(y + 4) = 0$, solve for x and y, if:
   a) $x = -3$ b) $x \neq 3$
   c) $y = -4$ d) $y \neq -4$
3. Solve for x by completing the square:
   a) $2x^2 + 7x - 4 = 0$
   b) $2x^2 - 3x - 3 = 0$ (leave answer in surd form)

Solutions

1. a) $x = 0$ or $x = \frac{5}{6}$

   b) $6x^2 - 5x - 6 = 0$
   $(2x - 3)(3x + 2) = 0$
   $x = 1\frac{1}{2}$ or $x = -\frac{2}{3}$

   c) $6x^2 + 11x + 5 = 15$
   $6x^2 + 11x - 10 = 0$
   $(2x + 5)(3x - 2) = 0$
   $x = -2\frac{1}{2}$ or $x = \frac{2}{3}$

   d) $8x^2 - 22x + 15 = 0$
   $(2x - 3)(4x - 5) = 0$
   $x = 1\frac{1}{2}$ or $x = 1\frac{1}{4}$

   e) $x^2 - 5x + 6 = 12$
   $x^2 - 5x - 6 = 0$
   $(x - 6)(x + 1) = 0$
   $x = 6$ or $x = -1$

   f) $x^2 - 7x - 8 = -18$
   $x^2 - 7x + 10 = 0$
   $(x - 5)(x - 2) = 0$
   $x = 5$ or $x = 2$

   g) $ax(x - 1) = 0$
   $x = 0$, if $a \neq 0$ or $x \in \mathbb{R}$, if $a = 0$ or $x = 1$

   h) $px^2 - 3px = 0$
   $px(x - 3) = 0$
   $x = 0$, if $p \neq 0$ or $x \in \mathbb{R}$, if $p = 0$ or $x = 3$

   i) $x^2 - x - 6 = x - 3$
   $x^2 - 2x - 3 = 0$
   $(x - 3)(x + 1) = 0$
   $x = 3$ or $x = -1$ or
   $(x - 3)(x + 2) - (x - 3) = 0$
   $(x - 3)(x + 2 - 1) = 0$ (take $(x - 3)$ out as a common factor)
   $(x - 3)(x + 1) = 0$
   $x = 3$ or $x = -1$

   j) $2(x^2 - 1) = 7x + 14 - 1$
   $2x^2 - 2 - 7x - 13 = 0$
   $2x^2 - 7x - 15 = 0$
   $(2x + 3)(x - 5) = 0$
   $x = 1\frac{1}{2}$ or $x = 5$

2. a) $y \in \mathbb{R}$ b) $y = -4$
   c) $x \in \mathbb{R}$ d) $x = 3$

3. a)

$$2x^2 + 7x = 4$$
$$x^2 + \frac{7}{2}x = 2$$
$$x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = 2 + \left(\frac{7}{4}\right)^2$$
$$\left(x + \frac{7}{4}\right)^2 = 2 + \frac{49}{16} = \frac{81}{16}$$
$$x + \frac{7}{4} = \pm\sqrt{\frac{81}{16}}$$
$$x = -\frac{7}{4} \pm \frac{9}{4}$$
$$x = -4 \text{ or } x = \frac{1}{2}$$

b)

$$2x^2 - 3x = 3$$
$$x^2 - \frac{3}{2}x = \frac{3}{2}$$
$$x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = \frac{3}{2} + \frac{9}{16}$$
$$\left(x - \frac{3}{4}\right)^2 = \frac{33}{16}$$
$$x - \frac{3}{4} = \pm\frac{\sqrt{33}}{4}$$
$$x = \frac{3}{4} \pm \frac{\sqrt{33}}{4}$$
$$x = \frac{3 \pm \sqrt{33}}{4}$$

Chapter 4 Numerical processes 1: Approximation and errors

Learning objectives

By the end of this chapter, the students should be able to:

1. Round off given values to the nearest ten, hundred, thousand, and so on, or to a given number of decimal places and/or significant figures.
2. Use rounded values to estimate the outcome of calculations.
3. Calculate the percentage error when using rounded values.
4. Decide on the degree of accuracy that is appropriate to given data that may have been rounded.

Teaching and learning materials

Students: 4-figure tables (provided on pages 237 and 248 of the Student's Book), calculator.
Teacher: Newspaper articles and reports that contain numerical data, population and other official data.

Areas of difficulty

Students may find it difficult to find the percentage error when they have to find the range of values of certain measurements. It would help if you could explain it as follows:

The biggest possible error when measuring is considered to be $\pm\frac{1}{2}$ of that unit.

Examples

1. **400 m to the nearest 0.1 of a m** means that the error would be $\frac{1}{2}$ of $\pm0.1 = \pm0.05$ and $400 - 0.05 \leq \text{length} < 400 + 0.05$, which gives $399.95 \leq \text{length} < 400.05$.
   $\%\text{ error} = \frac{\text{error}}{\text{measured length}} \times \frac{100}{1} = \pm\frac{0.05}{400} \times \frac{100}{1}$
   $= \pm0.0125\%$
2. **400 m to the nearest m** means that the error would be $\frac{1}{2}$ of $\pm 1 = \pm0.5$ and
   $400 - 0.5 \leq \text{length} < 400 + 0.5$,
   which gives $399.5 \leq \text{length} < 400.5$.
   $\%\text{ error} = \frac{\text{error}}{\text{measured length}} \times \frac{100}{1}$
   $= \pm\frac{0.5}{400} \times \frac{100}{1} = \pm0.125\%$
3. **400 m to the nearest 10 m** means that the error would be $\frac{1}{2}$ of $\pm1 = \pm5$ and
   $400 - 5 \leq \text{length} < 400 + 5$,
   which gives $395 \leq \text{length} < 405$.
   $\%\text{ error} = \frac{\text{error}}{\text{measured length}} \times \frac{100}{1}$
   $= \pm\frac{5}{400} \times \frac{100}{1} = \pm1.25\%$
4. **400 m correct to 1 s.f.** means that the error would be $\frac{1}{2}$ of $\pm100 = \pm50$ and
   $400 - 50 \leq \text{length} < 400 + 50$,
   which gives $350 \leq \text{length} < 450$.
   $\%\text{ error} = \frac{\text{error}}{\text{measured length}} \times \frac{100}{1}$
   $= \frac{\pm50}{400} \times \frac{100}{1} = \pm12.5\%$
5. **400 m to the nearest 2 m** means that the error would be $\frac{1}{2}$ of $\pm2 = \pm1$ and
   $400 - 1 \leq \text{length} < 400 + 1$,
   which gives $399 \leq \text{length} < 401$.
   $\%\text{ error} = \frac{\text{error}}{\text{measured length}} \times \frac{100}{1}$
   $= \pm\frac{1}{400} \times \frac{100}{1} = \pm0.25\%$
6. **4 000 m correct to 1 s.f.** means that the error would be $\frac{1}{2}$ of $\pm$ 1 000 = ±500 and
   $4\,000 - 500 \leq \text{length} < 4\,000 + 500$,
   which gives $3\,500 \leq \text{length} < 4\,500$.
   $\%\text{ error} = \frac{\text{error}}{\text{measured length}} \times \frac{100}{1}$
   $= \pm\frac{500}{4\,000} \times \frac{100}{1} = \pm12.5\%$

Chapter 5 Trigonometry 1: The sine rule

Learning objectives

By the end of this chapter, the students should be able to:

1. Determine the sine, cosine and tangent ratios of any angle between 0° and 360°.
2. Derive the sine rule.
3. Use the sine rule to solve triangles.
4. Apply the sine rule to real-life situations (such as bearings and distances, angles of elevation).

Teaching and learning materials

Students: 4-figure tables (provided on pages 237 and 248 of the Student's Book), calculator.
Teacher: Poster showing the sine rule and its relation to △ABC with sides *a*, *b*, *c*. Computer-assisted instructional materials where available. Chalk board compass and protractor.

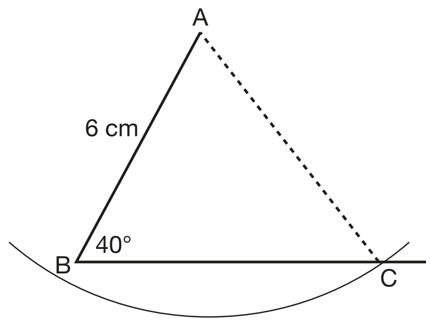
Areas of difficulty

- Students find applying the sine rule difficult, if the cosine and area rules are also taught at the same time. Students simply have to remember that, if there is a side and an angle opposite each other, they can use the sine rule.
- To make using the sine rule easy:
  - Use this version of the sine rule, if you have to work out a *side*: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
  - Use this version of the sine rule, if you have to work out an *angle*: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. Remember that the angle could also be an obtuse angle, because the sines of obtuse angles are positive.
- When you are given *side*, *side*, *angle* of a triangle that you have to solve, the possibility exists that there are two possible triangles. Students find it difficult to understand why this is the case.

  You could help them to understand this by actually constructing such a triangle. *Given: In any △ABC, ∠B = 40° and AB = 6 cm*. Now depending on the length of AC, you could get 1 or 2 triangles.

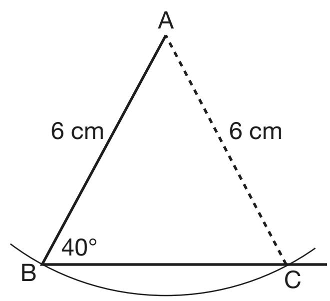
  To find point C, you measure the required length on a ruler with your compass. Then you put the sharp, metal point of the compass on A and chop off the required length through the line through B by drawing an arc with the pencil of your compass:

  a) If AC >AB, then the sketch will look something like this:



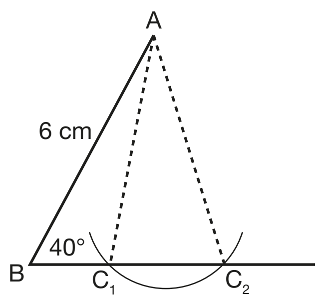
  The arc drawn will intersect the line at one point only. So, only one triangle is possible.

  b) If AC = AB, then the sketch will look something like this:



  The arc drawn will intersect the line at one point only. So, only one triangle is possible.

  c) AC < 6 cm (and not so short that the arc will not intersect the base line).

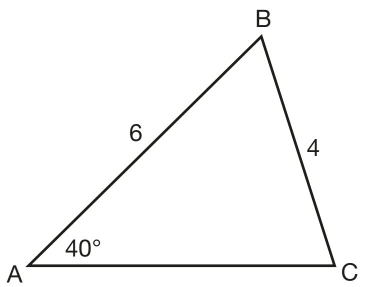


  The arc drawn will intersect the line in two places. So, two triangles are possible.

- This is probably why, when angle, side, side of a triangle are given, we say that it is an ambiguous case. If we do not know the length of the side opposite the given angle, we can have one triangle, two triangles or no triangle if the side is too short (that is when the sum of the two sides is less than the third side). Students do not have to construct the triangle. You only have to teach them to follow this procedure:
  1. Draw a sketch of the given triangle showing all the information given.
  2. If the side opposite the given angle is equal or longer than the side adjacent to the given angle, there is only one possible triangle.
  3. If the side opposite the given angle is shorter than the side adjacent to the given angle, there are two possible triangles.
  4. If the given angle is obtuse, the side opposite this angle is obviously longer than the side adjacent to the given angle. The longest side of a triangle is always opposite the biggest angle of the triangle. In a triangle, there is only one obtuse angle possible, because the sum of the angles of a triangle is equal to 180°.

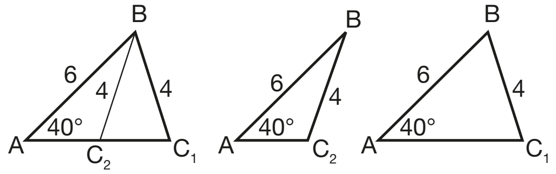
Supplementary worked examples

Solve $\triangle ABC$.



Solution

There are actually two possibilities for $\triangle ABC$: $\triangle ABC_2$ and $\triangle ABC_1$.



$\frac{\sin C}{6} = \frac{\sin 40°}{4}$ (put sin C on top, because you want to find the size of $\angle C$)

$\sin C = \frac{6\sin 40°}{4}$ (multiply both sides of the equation by 6)

$\therefore \angle C = 74.62°$ (On a calculator: 2ndF / shift, sin⁻¹ (6 sin 4 0 ° ÷ 4), =)

But this is the size of $\angle BC_1A$.

$\angle BC_2A = 180° - 74.62°$
$= 105.38°$ (the sum of the angles on straight line $AC_1 = 180°$)

$\angle ABC_1 = 180° - 40° - 74.62°$
$= 65.38°$ (sum $\angle$s $\triangle ABC_1 = 180°$)

$\angle ABC_2 = 180° - 40° - 105.38°$
$= 34.62$ (sum $\angle$s $\triangle ABC_2 = 180°$)

$\frac{AC_1}{\sin 65.38} = \frac{4}{\sin 40°}$

$AC_1 = \frac{4\sin 65.38}{\sin 40°} = 5.66$ (multiply both sides by sin 65.38°)

$\frac{AC_2}{\sin 34.62} = \frac{4}{\sin 40°}$

$AC_2 = \frac{4\sin 34.62}{\sin 40°} = 3.54$ (multiply both sides by sin 34.62°)

Chapter 6 Geometrical ratios

Learning objectives

By the end of this chapter, the students should be able to:

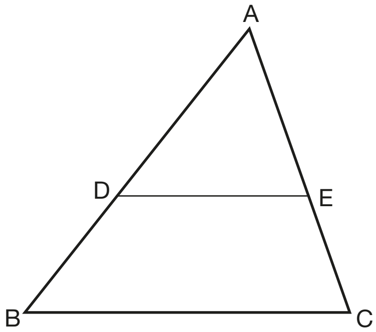
1. Recall and apply the following theorems:
   - equal intercept theorem
   - midpoint theorem
   - angle bisector theorem.
2. Apply the above theorems to:
   - division of line segments
   - proportional division of the sides of a triangle
   - ratios in geometrical figures.
3. Recall and use the properties of similar triangles to deduce lengths in similar shapes.

Teaching and learning materials

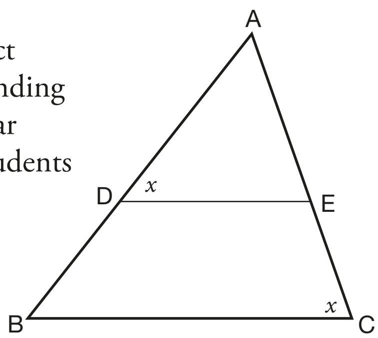
Students: Mathematical sets.
Teacher: Chalkboard protractor and compass, relevant posters and computer-assisted instructional software like Geometer's Sketchpad for example.

Areas of difficulty

- Writing ratios in the correct order. Emphasise for example:
  $\frac{AD}{DB} = \frac{AE}{EC}$
  and $\frac{DB}{AD} = \frac{EC}{AE}$
  $\frac{AD}{AB} = \frac{AE}{AC}$
  and $\frac{AB}{AD} = \frac{AC}{AE}$



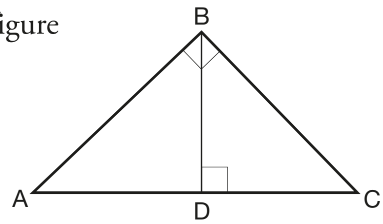
  The order of the ratio of the one side is also valid for the other side.
- Writing the correct ratios of corresponding sides of two similar triangles. Help students as follows:



1. Write the letters of the two similar triangles in the order of corresponding angles that are equal. For example:
   $\triangle ADE \,|||\, \triangle ACB$ (the $|||$ sign means similar)
2. Now write the ratios:
   $\triangle ADE \,|||\, \triangle ACB$
   $\frac{AD}{AC} = \frac{DE}{CB} = \frac{AE}{AB}$

Supplementary worked examples

a) Prove that the figure shown here has three similar triangles.



b) Prove the following:
   i) $AB^2 = AC \cdot AD$
   ii) $BC^2 = AC \cdot CD$

c) Use b) i) and ii) to prove Pythagoras's theorem.

Solution:

a) In $\triangle ABD$ and $\triangle ABC$:

$\angle BAD = \angle BAC$	(same angle)
$\angle BDA = \angle ABC = 90°$	(given)
$\angle ABD = \angle ACD$	(sum $\angle$s $\triangle$ = 180°)
$\therefore \triangle ABD \,|||\, \triangle ACB$	(3 $\angle$s of one $\triangle$ = to corr. $\angle$s of other $\triangle$)

In $\triangle CBD$ and $\triangle ABC$:

$\angle BCD = \angle BCA$	(same angle)
$\angle CDB = \angle CBA = 90°$	(given)
$\angle CBD = \angle BAC$	(sum $\angle$s $\triangle$ = 180°)
$\therefore \triangle CDB \,\vert\vert\vert\, \triangle CBA$	(3 $\angle$s of one $\triangle$ = to corr. $\angle$s of other $\triangle$)

In $\triangle ABD$ and $\triangle BCD$:

Let $\angle A = x$.

Then $\angle ABD = 90° - x$	(sum $\angle$s $\triangle$ = 180°)
$\therefore \angle CBD = 90° - (90° - x) = x$	
$\therefore \angle A = \angle CBD = x$	
$\angle BDA = \angle CDB = 90°$	(given)
$\angle ABD = \angle ACD$	(sum $\angle$s $\triangle$ = 180°)
$\therefore \triangle BAD \,\vert\vert\vert\, \triangle CBD$	(3 $\angle$s of one $\triangle$ = to corr. $\angle$s of other $\triangle$)

b) i) From $\triangle ABD \,|||\, \triangle ACB$:

$\frac{AB}{AC} = \frac{AD}{AB}$

$\therefore AB^2 = AC \cdot AD$

ii) From $\triangle CDB \,|||\, \triangle CBA$:

$\frac{BC}{AC} = \frac{CD}{BC}$

$\therefore BC^2 = AC \cdot CD$

c) $AB^2 + BC^2 = AC \cdot AD + AC \cdot CD$

$= AC(AD + CD)$	(take AC out as a common factor)
$= AC \cdot AC$	($AD + CD = AC$)
$= AC^2$	

Chapter 7 Algebraic processes 2: Simultaneous linear and quadratic equations

Learning objectives

By the end of this chapter, the students should be able to:

1. Solve simultaneous linear equations using elimination, substitution and graphical methods.
2. Solve simultaneous linear and quadratic equations using substitution and graphical methods.
3. Solve word problems leading to simultaneous linear equations and simultaneous linear and quadratic equations.

Teaching and learning materials

Students: Textbook and graph paper.
Teacher: Graph chalkboard or transparencies of graph paper and transparency pens, if an overhead projector is available. Wire that can bend and hold its form: to draw the curves of quadratic functions or plastic parabolic curves, if you have them.

Areas of difficulty

- When solving simultaneous linear equations, students experience these difficulties:
  1. They do not number the original equation, or the equations that they create, and the result is that they make unnecessary mistakes. Insist repeatedly that students number equations.
  2. They do not know whether to add or subtract equations to eliminate one of the variables. Teach them that:
     - If in the two equations the coefficients of the same variable are equal and also have the same sign, then they subtract the two equations from each other. For example:

       $2x - 3y = 10 \ldots$ ❶
       $2x + 5y = -14 \ldots$ ❷
       ❶ – ❷ : $-8y = 24$
       $\therefore y = -3$

       Or

       $-2x - 3y = 10 \ldots$ ❶
       $-2x + 5y = -14 \ldots$ ❷
       ❶ – ❷: $-8y = 24$
       $\therefore y = -3$
     - If the coefficients of the same variable are equal but have opposite signs, they must add the two equations. For example:

       $-2x - 3y = 10 \ldots$ ❶
       $2x + 5y = -14 \ldots$ ❷
       ❶ + ❷ : $2y = -4$
       $\therefore y = -2$
  3. When subtracting two equations, students sometimes forget that the signs of the equation at the bottom change. Again, emphasise that when subtracting ❶ from ❷ above, the left-hand side can also be written as $2x - 3y - (2x + 5y) = 2x - 3y - 2x - 5y = -8y$. The right-hand side can be written as $10 - (-14) = 10 + 14 = 24$.
  4. In equations such as $3x - 2y = 24$ and $4x - 9y = 36$, students must realise that to eliminate x, they must first get the LCM of 3 and 4, which is 12. Then the first equation must be multiplied by 4, because $4 \times 3 = 12$ and the second equation must be multiplied by 3, because $3 \times 4 = 12$:

     $3x - 2y = 24 \ldots$ ❶
     $4x - 9y = 36 \ldots$ ❷
     ❶ × 4: $12x - 8y = 96 \ldots$ ❸
     ❷ × 3: $12x - 27y = 108 \ldots$ ❹

     They could of course have used the LCM of 2 and 9, which is 18 and multiplied the first equation by 9 and the second by 2 to eliminate y.
- Students make mistakes if they substitute the value of the variable they found into an equation to find the value of the other variable.

The main reason for these errors is the fact that they do not use brackets for substitution. Let us say that they found that x, for example, is equal to -3. They have to substitute the value of x into $-3x + 2y = -4$ to find the value of y.

Then, to prevent errors, they should substitute the value of x like this:

$-3(-3) + 2y = -4$.

- When writing a two-digit number, students should realise the following:
  - If the tens digit is x and the units digit is y, they cannot write the number as xy.
  - In algebraic language xy, means $x \times y$.
  - Take an example such as 36 and explain that in our number system 36 means $3 \times 10 + 6$.
  - So a two digit number must be written as $10x + y$.
- When solving simultaneous linear and quadratic equations, students tend to make matters very difficult for themselves by choosing to make the variable (which will result in them working with a fraction) the subject of the linear equation. For example:

  If $3x + y = 10$ then solve for y and not x.

  If we solve for y, then $y = 10 - 3x$.

  On the other hand, if we solve for x, then $x = \frac{10 - y}{3}$.

  So, if the value of x has to substituted into a quadratic equation, the work will be much more difficult.

  So, insist that students always ask themselves what the easiest option is, and then make that variable the subject of the formula.
- Sometimes students find it difficult to factorise a quadratic equation. Teach them to use the quadratic formula if they find it difficult to get the factors.
- Students find it difficult to draw a perfect curve through the points on a parabola. Teach them to be very careful when drawing this curve.

  It would be ideal if you have a couple of plastic curves in your class.

  If not, you could always keep soft wire that can then be bent to fit all the points on the curve. Then a student could use a pencil and trace the curve made by the wire.
- If the graph of the quadratic function must be drawn to solve simultaneous equations (one linear and the other quadratic), the range of the x-values should be given so that the turning point of the graph or the quadratic function is included.
- Students almost always find word problems difficult. You could help them by using short, single sentences when setting test questions. You could also help them by leading them to find the equations by asking sub questions in your test or examination question.
  - In class, you must teach them how to recognise the variables. Then help them to translate the words sentence by sentence into Mathematics using these variables.
  - For problems where the area or perimeter of a plane figures is involved, insist that they make drawings to represent the situation.
  - The facts of some problems can also be represented in a table that makes it easier to understand. Number 14 of Exercise 7c can be represented in table form like this:

	Distance	Time	Speed
Run	8 km	$\frac{8}{x}$	x
Walk	2 km	$\frac{2}{y}$	y
Run	4 km	$\frac{4}{x}$	x
Walk	6 km	$\frac{6}{y}$	y

$\frac{8}{x} + \frac{2}{y} = \frac{5}{6}$

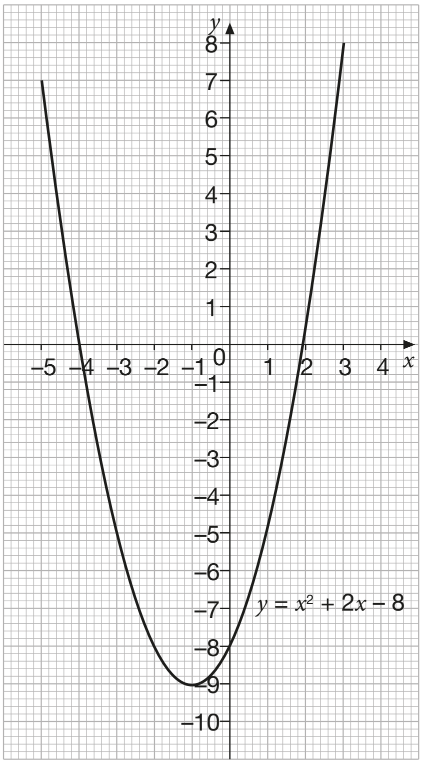
$\frac{4}{x} + \frac{6}{y} = \frac{5}{4}$

When solving a speed–time–distance problem, make sure that the speed is measured in the same unit as time. In the problem above, the times were in hours and minutes. In the equations, both times are in hours.

If your class struggles too much with word problems, do not spend unnecessary time on this work. Rather spend more time on the basics of this chapter where students can more easily earn marks to pass Mathematics.

Supplementary worked examples

1. Solve for x and y in these simultaneous equations:
   a) $\frac{81^{2x-3y}}{9} = 27$ and $\frac{64^{x-2y}}{16} = 1$
   b) $3^{x+2y} = 1$ and $2^{-x+5} = \left(\frac{1}{8}\right)^y$
   c) $(2x + 3y)(x - 2y) = 9$ and $x - 2y = 3$
2. Solve for x and y: i) algebraically and ii) graphically.
   a) $2x - 3y = 6$ and $4x - 6y = 12$
   b) $3x - 2y = -4$ and $6x - 4y = 8$
3. Use this graph to solve the following equations:



   a) $y = x^2 + 2x - 8$ and $y = -8$
   b) $y = x^2 + 2x - 8$ and $y = -9$
   c) $y = x^2 + 2x - 8 = 0$
   d) $x^2 + 2x - 12 = 0$
   e) $x^2 + 2x - 8 = -2x - 3$

Solutions

1. a)
$$(3^4)^{2x-3y} = 3^5$$
$$3^{8x-12y} = 3^5$$
$$8x - 12y = 5 \ldots ❶$$
$$(2^6)^{x-2y} = 2^4$$
$$2^{6x-12y} = 2^4$$
$$6x - 12y = 4 \ldots ❷$$
$$❶ - ❷: 2x = 1$$
$$\therefore x = \frac{1}{2}$$
Substitute $x = \frac{1}{2}$ into ❷:
$$6\left(\frac{1}{2}\right) - 12y = 4$$
$$-12y = 4 - 3$$
$$\therefore y = -\frac{1}{12}$$

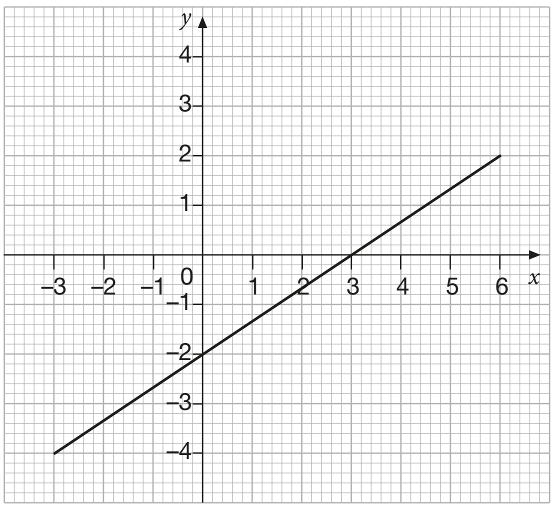
b)
$$3^{x+2y} = 3^0$$
$$x + 2y = 0 \ldots ❶$$
$$2^{-x+5} = (2^{-3})^y$$
$$2^{-x+5} = 2^{-3y}$$
$$-x + 5 = -3y$$
$$-x + 3y = -5 \ldots ❷$$
$$❶ + ❷: 5y = -5$$
$$\therefore y = -1$$
Substitute $y = -1$ into ❶:
$$x - 2 = 0$$
$$\therefore x = 2$$

c)
$$(2x + 3y)(3) = 9$$
$$2x + 3y = 3 \ldots ❶$$
$$x - 2y = 3 \ldots ❷$$
$$❷ \times 2: 2x - 4y = 6 \ldots ❸$$
$$❶ - ❸: 7y = -3$$
$$\therefore y = -\frac{3}{7}$$
Substitute $y = -\frac{3}{7}$ into ❷:
$$x - 2\left(-\frac{3}{7}\right) = 3$$
$$x + \frac{6}{7} = 3$$
$$\therefore x = 2\tfrac{1}{7}$$

2. a) i)
$$2x - 3y = 6 \ldots ❶$$
$$4x - 6y = 12 \ldots ❷$$
$$❷ \div 2: 2x - 3y = 6$$
Answer: $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

ii) $-3y = -2x + 6 \Rightarrow y = \frac{2}{3}x - 2$

x	-3	0	3
$y = \frac{2}{3}x - 2$	-4	-2	0



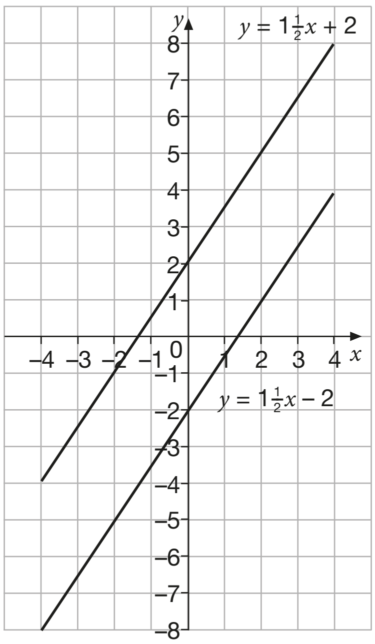
The two equations are actually just multiples of each other.
So, they are represented by the same graph.
So, all the real values of x and all the real values of y will satisfy both equations simultaneously.

b) i) $3x - 2y = -4$ … ❶
$6x - 4y = 8$ … ❷
❶ × 2: $6x - 4y = -3$ … ❸
However, $6x - 4y$ cannot be equal to -8 and 8. So, there is no solution.

ii) $-2y = -3x - 4$
$y = 1\frac{1}{2}x + 2$
$-4y = -6x + 8$
$\therefore y = 1\frac{1}{2}x - 2$

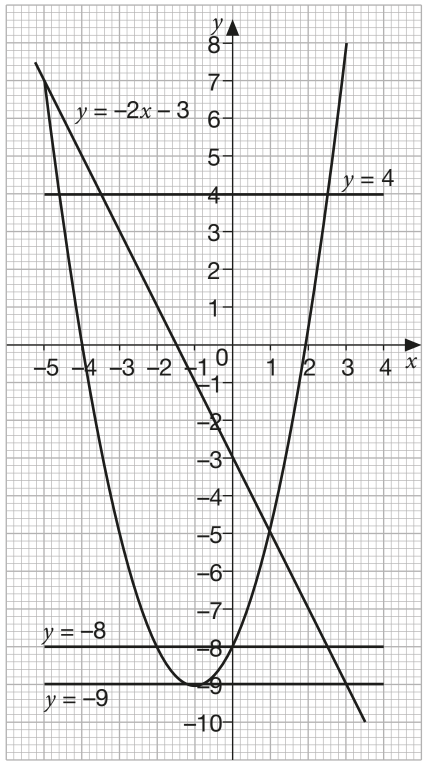
x	-2	0	2
$y = 1\frac{1}{2}x + 2$	-1	2	5

x	-4	0	4
$y = 1\frac{1}{2}x - 2$	-8	-2	4



In the graph above, you can see that the two graphs will never intersect because they are parallel. So, there is no solution, or no point that lies on both graphs, or no x- and y-values that satisfy the two equations simultaneously.

3. a) Draw $y = -8$:



Read from graph: Therefore, $x = -2$ or $x = 0$.

b) Draw $y = -9$.
Read from graph: Therefore, $x = -1$.

c) $y = 0$ is the x-axis: Therefore, $x = -4$ or $x = 2$.

d) $x^2 + 2x - 12 + 4 = 4$
$x^2 + 2x - 8 = 4$
Draw $y = 4$. Therefore, $x = -4.6$ or $x = 2.6$.

e) Draw the graph of $y = -2x - 3$:
Therefore, $x = -5$ or $x = 1$ and $y = 7$ or $y = -5$.

Chapter 8 Statistics 1: Measures of central tendency

Learning objectives

By the end of this chapter, the students should be able to:

1. Calculate and interpret the mean, median and mode of ungrouped data.
2. Apply the calculation of means to average rates and mixtures.

Teaching and learning materials

Students: Textbook and exercise book.
Teacher: Computer with appropriate software, if available.

Areas of difficulty

- When calculating average rates, students tend to add up the rates and then divide the answer by the number of rates.
  Explain that the average of a rate is always the total of the quantities involved divided by the total of the "per" unit involved. The "per" unit could be km^2, hour, kg, and so on. For example:
  - The average speed in km/h is equal to the total distance in km divided by the total time in hours.
  - The average profit per day is equal to the total amount of money earned during a certain period divided by the number of days of the period.
  - The average density in cm^3/g of three different substances is the total volume in cm^3 divided by the total mass in g.
- Students could find problems like Example 9 of this chapter very difficult. You could perhaps explain this example as follows:
  Now you want to break even by not having a loss or a profit.
  - The first tea: On each kg at ₦760, the loss is ₦50.
  - The second tea: On each kg at ₦840, the gain (or profit) is ₦30.
    Now the LCM of 50 and 30 is 150.
  - For the first tea, $\frac{150}{50} = 3$. So for 3 kg, of the first tea, the loss is ₦150.
  - For the second tea, $\frac{150}{30} = 5$. So for 5 kg, of the first tea, the gain is ₦150.

  So, the first tea must be mixed with the second in a ratio of 3 : 5. Now, you can ask any amount per kg for the mixture, because the profit or loss according to the original costs does not play a role.

Chapter 9 Trigonometry 2: The cosine rule

Learning objectives

By the end of this chapter, the students should be able to:
1. Derive the cosine rule.
2. Use the cosine rule to calculate the lengths and angles in triangles.
3. Use the sine and cosine rules to solve triangles.
4. Apply trigonometry to solve real-life situations (such as bearings and distances).

Teaching and learning materials

Students: Textbook, exercise book and 4-figure tables (provided on pages 236 and 247 of the Student's Book) and a scientific calculator.
Teacher: Poster showing the cosine rule and its relation to any △ABC with sides *a*, *b*, and *c*. Computer-assisted instructional materials where available.

Areas of difficulty

- Students cannot remember the cosine rule or how to apply it when they have to work out a side of a triangle. Teach them that, if in a triangle there are no sides and angles opposite each other, the cosine rule must be used. If two sides with the angle between them are given, then they can work out the side opposite the angle by applying the cosine rule as follows:
  (Opposite side)^2 = (adjacent side)^2 + (other adjacent side)^2 – 2(adjacent side)(other adjacent side)(cos of ∠ between the two sides)
  Also emphasise that they now have (Opposite side)^2 and they must first get the square root of their answer to get the length of the side. So in the beginning, students would, for example, have written $b^2 = \ldots$, but for the final answer they should not forget to now write $b = \ldots$ for the square root of the first answer.
- Students may find it difficult to work with calculators to work out a side of the triangle. They could also work out the answer of Example 3 like this if they use a scientific calculator:
  8 . 4 4, 4 x^2 × 7 . 9 2, x^2 − 2 × 8 . 4 4 × 7 . 9 2 × cos 1 5 1 . 4 = √ = … . They do not have to use the memory of the calculator.
- If three sides of a triangle are given, the cosine rule is always used, but students often do not know how to apply the cosine rule correctly. Teach them to do the following:
  Choose any angle. Then write:
  cos of that angle
  $$= \frac{(\text{adjacent side})^2 + (\text{other adjacent side})^2 - (\text{opposite side})^2}{2 \times (\text{adjacent side})(\text{other adjacent side})}$$
- Students may find it difficult to work the angle out with a calculator. They can also work out the answer of Example 6 like this:
  4 . 1, x^2 + 5 . 9, x^2 − 7 . 7, x^2 = ÷ (2 × 4 . 1 × 5 . 9) = 2ndF (or shift), $\cos^{-1}$ = … . They do not have to use the memory of the calculator.
  In all these calculations, students must make sure that their calculator is set on degrees and not on radians or grads.
- After a side of a triangle is calculated using the cosine rule, an angle must be calculated using the sine rule. In this case, a side and a side and an angle not between the two sides are given.

- In Chapter 5, you taught the students that this combination of measurements of a triangle might result in the possibility of two triangles, if the side opposite the given angle is shorter than the side adjacent to the given angle.
- To prevent this happening and the students becoming confused, teach them to always work out the angle opposite the shortest side first.
- When solving problems involving bearings and distances, students may find it difficult to make the sketches. Teach them to always start with the N–S and W–E cross and to work from there.

Chapter 10 Algebraic processes 3: Linear inequalities

Learning objectives

By the end of this chapter, the students should be able to:

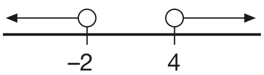
1. Represent inequalities in one variable on a number line.
2. Solve inequalities in one variable.
3. Represent inequalities in two variables on a Cartesian graph.
4. Solve linear inequalities in two variables showing required solution sets within an unshaded region on a Cartesian graph.
5. Solve real-life problems involving restrictions using graphical methods (that is, linear programming).
6. Use linear programming to determine maximum and minimum values within a given set of restrictions.

Teaching and learning materials

Students: Textbook, exercise book, graph paper and mathematical sets.
Teacher: Graph chalk board, board instruments (ruler and set square for parallel lines) or transparency graph paper, an overhead projector, transparency pens and a ruler, if available.

Areas of difficulty

- Students tend to write the inequality shown in this diagram:

  as $-2 > x > 4$: if you read it, it would read: x is smaller than -2 **and** bigger than 4. There is no number that is smaller than -2 **and** greater than 4.
  Teach students that there are two parts and that the above inequality is actually the result of the union of two inequalities:
  $\{x; x < -2, x \in \mathbb{R}\} \cup \{x: x > 4, x \in \mathbb{R}\}$ and that they should always write: $x < -2$ **or** $x > 4$.
  The word "**or**" represents the "**union**" of two sets.
- Some students will find drawing straight-line graphs and determining their equations very difficult if they have not done drawing and finding the equations of straight lines recently. To prevent this, Chapter 16 can be done first before linear inequalities in two variables and linear programming is done.
- Students tend to not reverse the inequality symbol when they divide or multiply by a negative value.
  - Explain to the students that, when an equation is solved, we add or subtract the same quantity or multiply or divide by the same quantity both sides of the equation.
  - To explain this, you can now do this, for example:
    $4 > 2$
    Add 2 to both sides: $6 > 4$ *Which is still true.*
    Add -2 to both sides: $2 > 0$ *Which is still true.*
    Subtract 3 from both sides:
    $1 > -1$ *Which is still true.*
    Subtract -3 from both sides: $4 - (-3) = 7$ and $2 - (-3) = 5$ and $7 > 5$ *Which is still true.*
    Multiply both sides by 2:
    $8 > 4$ *Which is still true.*
    Multiply both sides by -2:
    $-8 > -4$ *Which is not true.*
    For this to be true, we must write: $-8 < -4$.
    So, the inequality sign must reverse.

Divide both sides by 2:

$2 > 1$ *Which is still true.*

Divide both sides by −2:

$-2 > -1$ *Which is not true.*

For this to be true, we must write: $-2 < -1$.

So, the inequality sign must reverse.

- Students do not know which inequality sign to use if certain words are used. Below is a summary of some of the words used:
  - *At least* means at the minimum or not less than or bigger than or equal. $\geq$
  - *Not more than* means less than or equal. $\leq$
  - *Up to* means at the most or the maximum or less than or equal. $\leq$
  - *The minimum* means bigger than or equal. $\geq$
  - *Not less than* means it must be more than or equal. $\geq$
- Students may find it difficult to work with the objective function as a family of parallel lines.

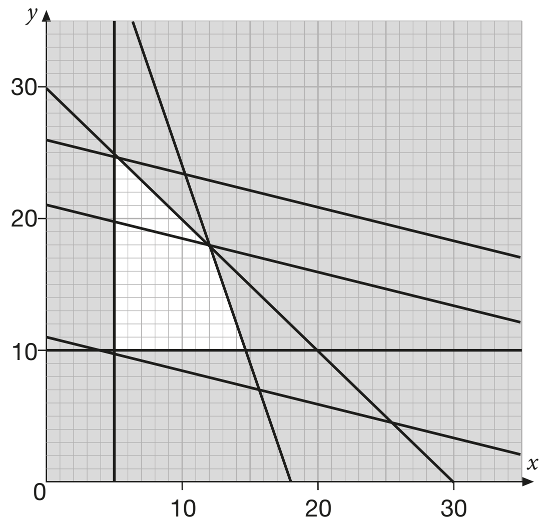
  They do not understand that to maximise they have to move the line as far as possible up along the y-axis, while still keeping the same gradient.

  If the objective function is profit (P) in $P = x + 4y$, for example, they need to make y the subject of the equation: $-4y = x - P$

  $y = -\frac{1}{4}x + \frac{1}{4}P$.

  So, because P is part of the y-intercept and we want the profit (P) to be as big as possible, the line with the gradient $-\frac{1}{4}$ must be moved up as far as possible, while still satisfying the restrictions of the situation.

  So, part of the line must at least still touch one of the points of the lines that make up the boundaries of the region that contains the possible values of x and y. **This region is called the feasible region.**

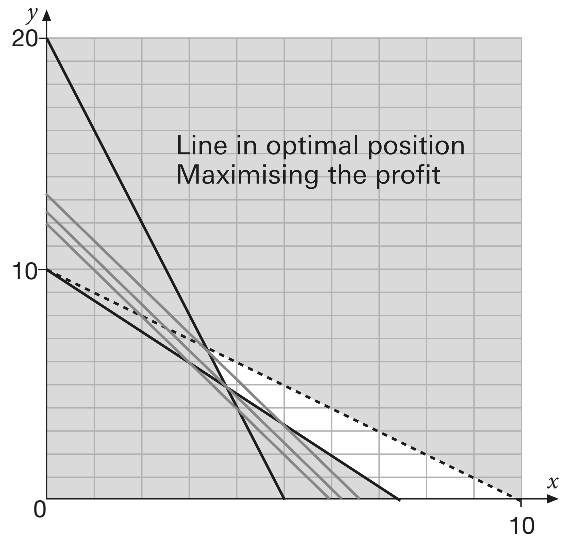


The opposite happens, if one wants to minimise cost, for example:

Here the cost, $C = 20x + 10y$. Making y the subject of the formula, one gets $y = -2x + 0.1C$.

Now the y-intercept, which is part of the cost, must be as low as possible while still touching at least one point that is part of the feasible region.

The line in the middle below shows the optimal position of the objective function.



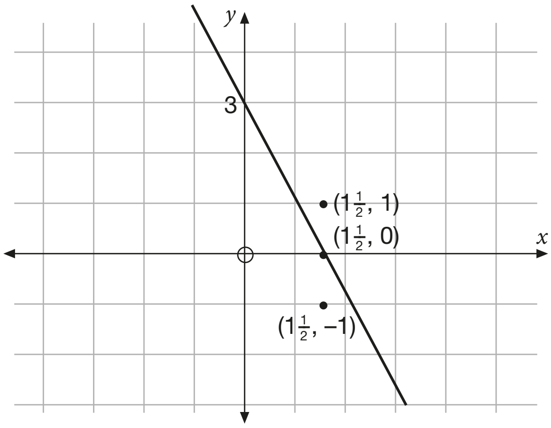
Supplementary worked examples

- When inequalities with two variables have to be represented, you can explain where to shade $y < 3 - 2x$, or where to shade $y \geq 3 - 2x$ by doing the following:

  Draw the graph of $y = 3 - 2x$ by joining its x- and y-intercepts.

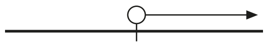
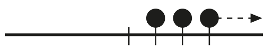
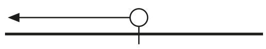
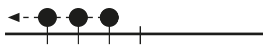
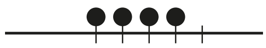
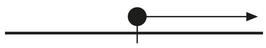
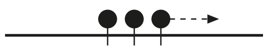
Then choose any point on the line for example $(1\frac{1}{2}, 0)$.

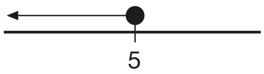
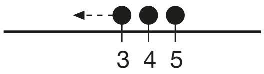
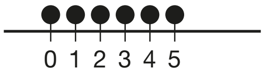
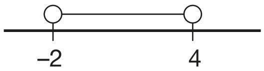
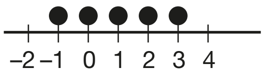
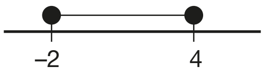
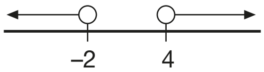
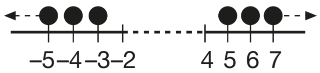
Now, choose a point above the line, for example, $(1\frac{1}{2}, 1)$ and a point below the line, for example, $(1\frac{1}{2}, -1)$.



The point on the line satisfies the equation:
$y = 3 - 2x = 3 - 2\left(1\frac{1}{2}\right) = 3 - 3 = 0$.
So, all the points on the line satisfy the equation $y = 3 - 2x$.
Substitute the point above the line, and you will get $3 - 2x = 0$ and $y = 1$.
Thus, $1 > 0$ and the points of the part of the Cartesian plane above the line of $y = 3 - 2x$ satisfies the inequality $y > 3 - 2x$.
Substitute the point below the line, and you will get $3 - 2x = 0$ and $y = -1$.
Therefore, $-1 < 0$ and the points of the part of the Cartesian plane below the line of $y = 3 - 2x$ satisfy the inequality $y < 3 - 2x$.
You can now draw the conclusion that when y is the subject of the formula, the region above the line satisfies the inequality $y > mx + c$, and the region below the line satisfies the inequality $y < mx + c$.
Now emphasise that, if you have $\geq$ or $\leq$, the line is solid, because the point are included.
If the inequality is only $<$ or $>$, the points on the line are not included and is shown as a broken line. This method will only work when the inequality is written in the form where only y is the subject of the formula.

- Students need to know that, if an inequality is represented by means of a solid line, it means that the numbers represented are real numbers.
  - If the numbers are
    Natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$ or
    Counting numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ or
    Integers, $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, \ldots\}$, then separate numbers as dots must be shown on the number line.
  - Here is a summary:

Inequality written mathematically	Inequality represented graphically	Inequality in words and/or clarifying notes
$x > 5$; $x \in \mathbb{R}$	 5	x is greater than 5 (if the number is not included, the circle is left open). The arrow shows that the numbers go on.
$x > 5$; $x \in \mathbb{Z}$ or $x \in \mathbb{N}$	 5 6 7 8	The broken-line arrow shows that the numbers go on.
$x < 5$; $x \in \mathbb{R}$	 5	x is less than 5, (if the number is not included, the circle is left open). The arrow shows that the numbers go on.
$x < 5$; $x \in \mathbb{Z}$	 2 3 4 5	Separate integers are shown with dots and the broken-line arrow shows that the numbers go on.
$x < 5$; $x \in \mathbb{N}$	 1 2 3 4 5	Natural numbers start at 1 and separate Natural numbers are shown by dots.
$x \geq 5$	 5	x is greater than or equal to 5, (if the number is included, the circle is shaded).
$x \geq 5$; $x \in \mathbb{Z}$ or $x \in \mathbb{N}$	 5 6 7	Separate Integers or Natural numbers are shown by dots and the broken-line arrow shows that the numbers go on.

$x \leq 5; x \in \mathbb{R}$		x is less than or equal to 5, (if the number is included, the circle is shaded).
$x \leq 5; x \in \mathbb{Z}$		Separate Integers are shown by dots and the broken-line arrow shows that the numbers go on.
$x \leq 5; x \in \mathbb{N}_0$		Counting numbers start at 0 and are shown by $\mathbb{N}_0$.
$-2 < x < 4; x \in \mathbb{R}$		x is greater than –2 **and** less than 5 or x is between –2 and 4 or $\{x \mid x > \ ; x \in \mathbb{R}\} \cap \{x \mid x < 4; x \in \mathbb{R}\}$.
$-2 < x < 4; x \in \mathbb{Z}$		x is greater than –2 **and** less than 5 or x is between –2 and 4 or $\{x \mid x > 2; x \in \mathbb{Z}\} \cap \{x \mid x < 4; x \in \mathbb{Z}\}$.
$-2 \leq x \leq 4; x \in \mathbb{R}$		x is greater than or equal to –2 **and** less or equal to 5, (the shaded circles show that –2 and 5 are included) or x is between –2 and 4; –2 and 4 included or $\{x \mid x \geq 2; x \in \mathbb{R}\} \cap \{x \mid x \leq 4; x \in \mathbb{R}\}$.
$x < -2$ or $x > 4; x \in \mathbb{R}$		x is less than –2 or x is greater than 4 or $\{x \mid x < 2; x \in \mathbb{R}\} \cup \{x \mid x > 4; x \in \mathbb{R}\}$.
$x < -2$ or $x > 4; x \in \mathbb{Z}$		It is not necessary to show all the numbers in between.

Chapter 11 Statistics 2: Probability

Learning objectives

By the end of this chapter, the students should be able to:

1. Use the language of probability, including applications to set language, to describe and evaluate events involving chance.
2. Define and use experimental probabilities to estimate problems involving chance.
3. Define and use theoretical probabilities to calculate problems involving chance.
4. Solve problems involving combined probabilities, particularly those relating to mutually exclusive events and/or independent evens.
5. Illustrate probability spaces by means of outcome tables, tree diagram and Venn diagrams, and use them to solve probability problems.

Teaching and learning materials

Students: Textbook, exercise book, graph paper, coins, and a simple die made from a hexagonal pencil or ballpoint pen.

Teacher: Class sets of coins, dice, counters, playing cards, graph board.

Glossary of terms

Outcome is a possible result of an experiment. Each possible **outcome** of a particular experiment is unique. (Only one **outcome** will occur on each trial of the experiment.)

Certain event is something that is certain to happen and its probability is always equal to 1. For example, the probability that the sun rises in the East in London is 1.

Impossible event is something that will never happen and is always equal to 0. For example, the probability that the sun rises in the West in London is 0.

Theoretical probability is probability based on logical reasoning written as $\frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}}$

Disjoint events are also mutually exclusive events. If two **events** are **disjoint**, then the probability of them both occurring at the same time is 0.

Areas of difficulty

- Words like "at most" and "at least" could give problems. Explain as follows:
  Say you are choosing 3 cards from a pack of cards and you want to know what the probability is of choosing spades.
  Then choosing **at most** 3 spades means that you could choose 3 spades, 2 spades, 1 spade or 0 spades.
  Choosing at least 2 spades means that you can choose 2 spades or 3 spades.
- If a choice is made and the object is not put back, students tend to forget that the total number of objects is now 1 less and the number of the objects chosen and not put back is also 1 less. The quantity of the other kinds of objects stays the same.
  For example, you have 3 black and 4 white balls in a bag.
  If you take out one black ball and you do not put it back, there are 2 black balls left and the total number of balls now is 6.
  The number of white balls stays the same. So the probability that a black ball is taken out again is $\frac{2}{6} = \frac{1}{3}$, and the probability that a white ball is taken out at the second draw is $\frac{4}{6} = \frac{2}{3}$.

Chapter 12 Circle geometry: Tangents

Learning objectives

By the end of this chapter, the students should be able to:

1. Define and apply the properties of a tangent of a circle.
2. Recall, use and apply the following tangent properties to solve circle problems:
   - tangents from an external point are equal
   - the alternate segment theorem
   - angle between tangent and radius = 90°.

Teaching and learning materials

Students: Textbook, exercise book.
Teacher: Posters, cardboard models, chalkboard instruments (especially a protractor and a compass), computer instructional materials where available.

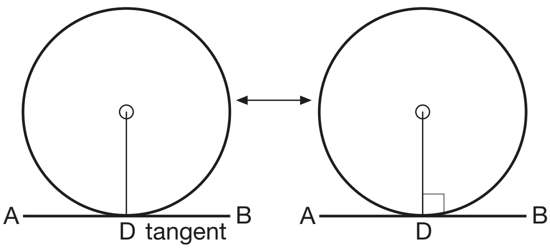
Glossary of terms

Secant of a circle is a line that intersects the circle in two places or it has two points in common with the circle.

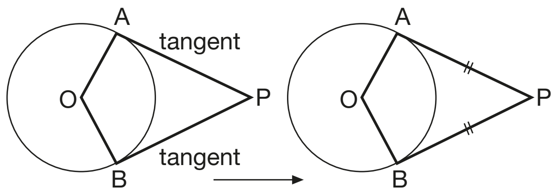
Tangent of a circle is a line that intersects the circle in only one place. We can also say that it has only one point in common with the circle.

Areas of difficulty

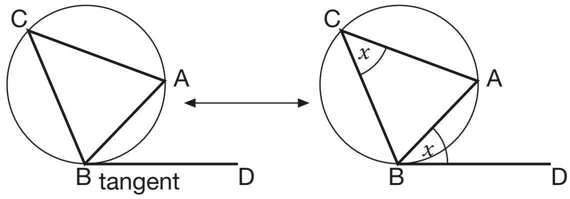
- Students tend to forget theorems and deductions made from them.
  - Students must always give a reason for each statement that they make which is based on some or other theorem and they either do not give the reason or do not know how.
  - To help them you could give them summaries of the theorems and suggestions of the reasons they can give. Below are suggestions:



Reason: Radius ⊥ tangent



Reason: Tangents from same point P are =



Reason: Angle between tangent BD and chord BA = **angle on BA**

- Students find it difficult to recognise the angle in the alternate or opposite segment which is equal to the angle between the tangent and the chord.
  - Teach them that it is always the angle on the chord and the angle that touches the circumference of the circle.
  - If they still cannot recognise this, let them put their forefinger and thumb on the chord that shares the point of contact with the tangent and follow the two lines at its ends to the angle on the circumference of the circle.

Chapter 13 Vectors

Learning objectives

By the end of this chapter, the students should be able to:

1. Define and distinguish between displacement, equivalent and position vectors.
2. Calculate the magnitude and direction of a vector.
3. Find the sum and difference of vectors.
4. Define a null (zero) vector.
5. Multiply a vector by a scalar.
6. Resolve a vector into components.
7. Solve vector algebra problems.

Teaching and learning materials

Students: Textbook, graph exercise book, ruler and pencil.

Teacher: Graph board or graph paper transparencies, a ruler, transparency pens and an overhead projector, if available.

Glossary of terms

Base vectors are the vectors $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and are in the positive direction of the x- and the y-axes respectively.

Components of vectors are the scalar multiples of i and j.

Equivalent vectors are vectors with equal magnitudes and directions, but they have different points of application. The symbol for equivalent vectors is ≡.

Position vector is a vector that gives the displacement of a point relative to the origin

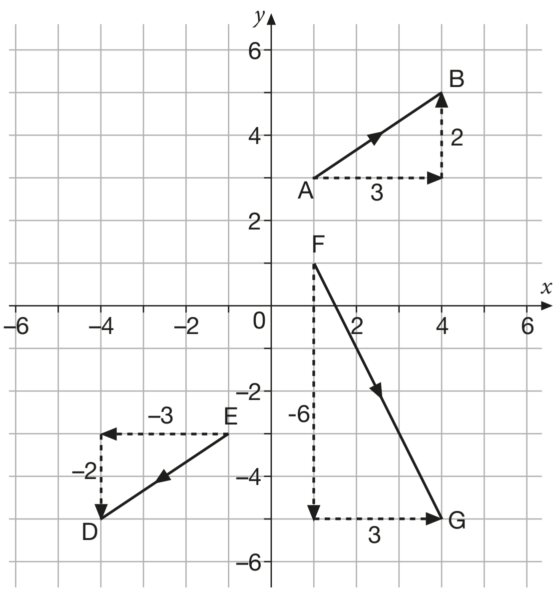
Vector sum is a single displacement that is equivalent to the separate displacements of two or more vectors.

Magnitude or modulus of a vector is the length of the line segment that represents the specific vector **AB** $= \begin{pmatrix} x \\ y \end{pmatrix}$ and is $|AB| = \sqrt{x^2 + y^2}$.

Scalar is a quantity with only size and no direction for example, mass, volume, thickness, and so on.

Areas of difficulty

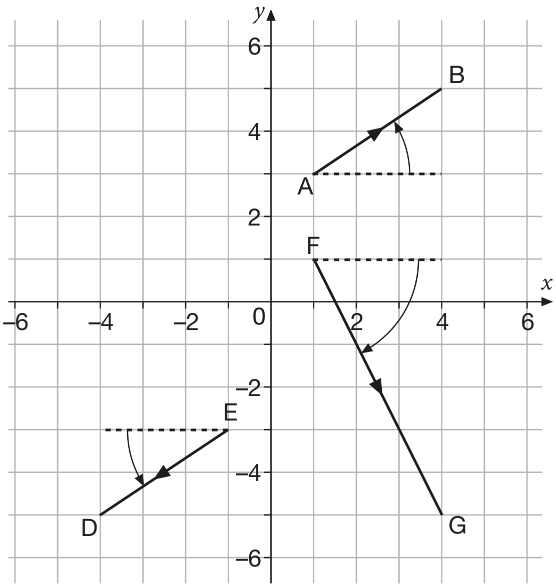
- Students must remember that, if they represent a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on squared paper or graph paper:
  - A positive x component is a horizontal displacement to the right.
  - A negative x component is a horizontal displacement to the left.
  - A positive y component is a vertical displacement upwards.
  - A negative y component is a vertical displacement downwards.



- Students must remember that if a figure is enlarged so that its sides are x times longer, then its area is x^2 times bigger.

  The reason is that if an area is calculated, two lengths are used and if each length is, for example, 2 times longer, the corresponding area is 2×2 bigger.
  Say we have a triangle with its base = 1 unit and its height = 4 units.
  Now, if we enlarge this triangle so that all its **sides are 3 times as long**, then the base = 3 units and the height = 12 units.
  - The area of the original triangle $= \frac{1}{2}(1)(4) = 2$ units^2.
  - The **area of the enlarged triangle** $= \frac{1}{2}(3)(12) = 18$ units^2, which is **9 (or 3^2)** times bigger than the area of the original triangle.

- If the direction of a vector is measured, it is always measured anti-clockwise from the horizontal.



Chapter 14 Statistics 3: Grouped data

Learning objectives

By the end of this chapter, the students should be able to:

1. Present and interpret grouped data in frequency tables, class intervals, histogram and frequency polygons.
2. Calculate, illustrate and interpret the mean, median and mode of grouped data.
3. Construct cumulative frequency tables.
4. Draw cumulative frequency curves and use them to deduce quartiles and medians.

Teaching and learning materials

Students: Writing materials, exercise book, textbook, graph paper, a soft wire that can be used to draw the curve of the cumulative frequency curves and a ruler.

Teacher: Graph chalkboard; computer with appropriate software if available; copies of graph paper on transparencies, transparency pen and an overhead projector, if available.

Areas of difficulty

- Students make the mistake of not starting and ending a frequency polygon on the horizontal axis. Teach them that they always start with the class midpoint of the class before the first class (even if it is a negative value) and end with class midpoint of the class after the last class.
- If the scale of the cumulative frequency curve is too small, accurate readings of the median, first quartile, third quartile, and so on cannot be taken. Students should make the scale for this curve as big as possible.
- Make sure that students know that for the ogive or cumulative frequency curve, they plot the upper class boundaries (on the horizontal axis) against the cumulative frequency (on the vertical axis).
- The accuracy of the readings from the ogive also depends on how accurate the curve is drawn. Students can use a soft wire and bend it so that it passes through all the plotted points. They can then trace the curve of the wire on the graph paper to draw an accurate curve.

Chapter 15 Transformation geometry

Learning objectives

By the end of this chapter, the students should be able to:

1. Explain what is meant by geometrical transformations, congruencies and enlargements.
2. Define and apply the properties of translation, reflections and rotation in a plane.
3. Use reflection and rotation to determine properties of plane shapes.
4. Determine and use the scale factor of an enlargement.
5. Construct enlargements of given shapes.
6. Locate the centre of an enlargement.
7. Solve problems involving combined transformations.

Teaching and learning materials

Students: Textbook, graph exercise book, ruler and pencil.

Teacher: Graph board or graph paper transparencies, a ruler, transparency pens and an overhead projector, if available.

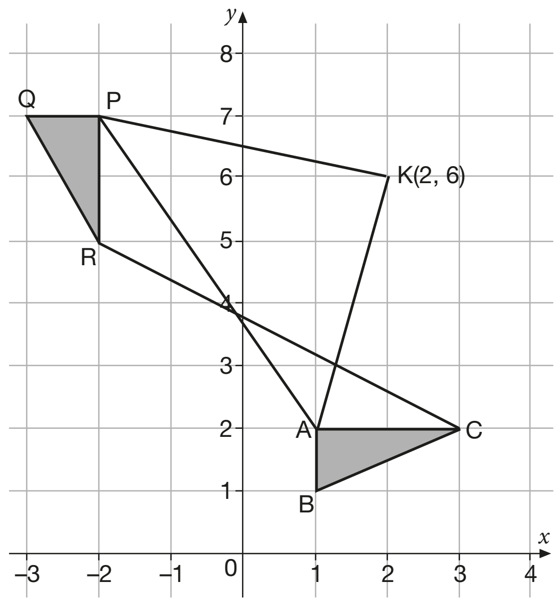
Glossary of terms

Congruencies (in transformation geometry) mean that the shape and its image have identical dimensions. Translation, reflection and rotation produce images congruent to the original figure.

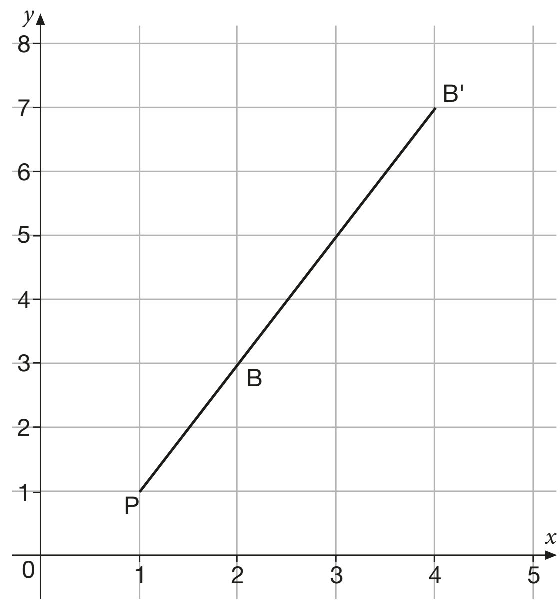
Displacement vector: displacement is the shortest distance from the initial to the final position of a point P. It is, therefore, the length of an imaginary straight path; and it is usually not the actual path travelled by P. A displacement vector represents the length and direction of this imaginary straight path. If the displacement vector is $\binom{a}{b}$, for example, then all the points (x, y) of a quadrilateral will change to $(x + a, y + b)$, or we can also say that all the position vectors $\binom{x}{y} \rightarrow \binom{x+a}{y+b}$ (where $\rightarrow$ means "change to").

Areas of difficulty

- When you want to explain how the angle of rotation is determined you could tell the class the following:
  - Join A of the original figure, to K, the centre of rotation.
  - Join P, the corresponding point of the image to K, the centre of rotation.
  - Measure the angle PKA. In this case, $\angle PKA = 90°$ and the rotational direction from A to P is anti-clockwise.

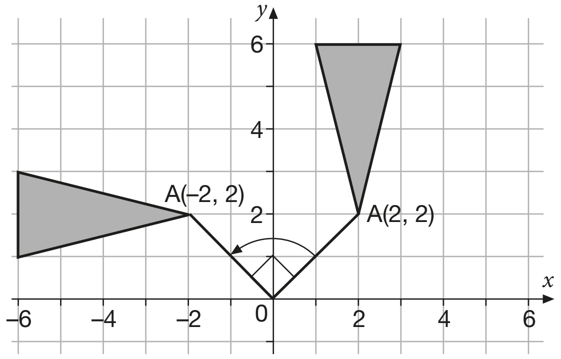


- To find the centre of enlargement can be tricky if students do not know what the images of the specific points are, because to find the centre of enlargement a point must be joined to its image and another point must be joined to its image.
  The centre of enlargement then is the point where these two line segments intersect. Usually it can be seen from the form of the figure what the images of the different points are (otherwise it must be given) because different images would result in different enlargements as can be seen from 2. d) of Exercise 15f.
- The scale factor is always measured from the centre of enlargement. If the scale factor is 3 and the centre of enlargement is P, then $\frac{PB'}{PB} = \frac{3}{1}$ and you should draw PB' three times as long as PB (BB' = 2PB).



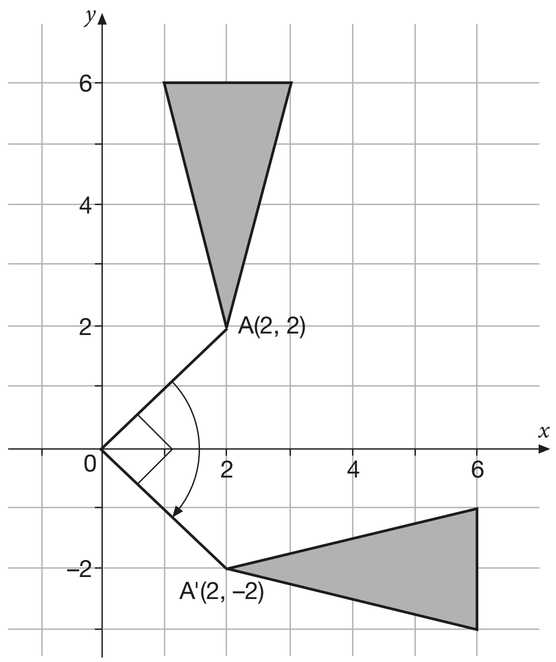
Supplementary worked examples

1. When a figure is rotated around the origin, specific rules can be deduced. Students should find these rules out for themselves.



They should, for example draw a triangle A(2, 2), B(3, 6) and C(3, 1) on squared paper or graph paper.

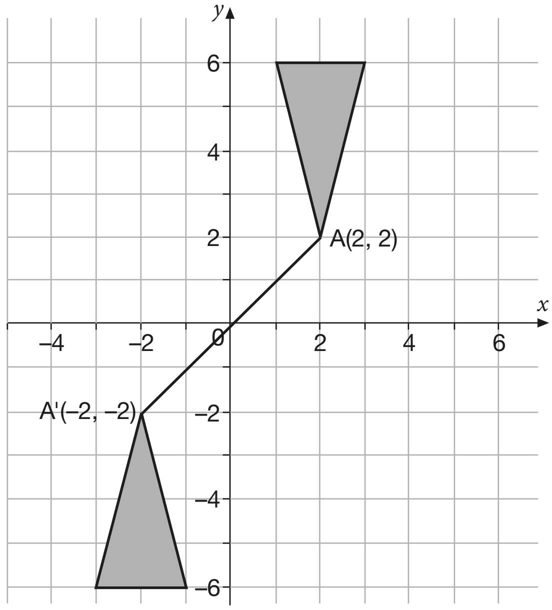
- Then they must join A, to the origin, measure an angle of **90° anti-clockwise** and draw a line segment equal in length to OA to find A'.
- They will then see that A' is the point (–2, 2). The same process could be repeated to find B' and C':
  - **A general rule can, therefore, be deduced. It states that if any point is rotated through 90° anticlockwise around the origin, then $(x, y) \rightarrow (-x, y)$.**
  - **This rule also applies for a rotation of 270° clockwise around the origin, because it will end in the same position as the figure above.**



Students then could take the same triangle again and measure an angle of 90° clockwise around the origin and follow the same procedure as described above:

- They will then see that A' is the point (2, –2). The same process could be repeated to find B' and C':
  - **A general rule can, therefore, be deduced. It states that if any point is rotated clockwise through 90°around the origin, then $(x, y) \rightarrow (x, -y)$.**

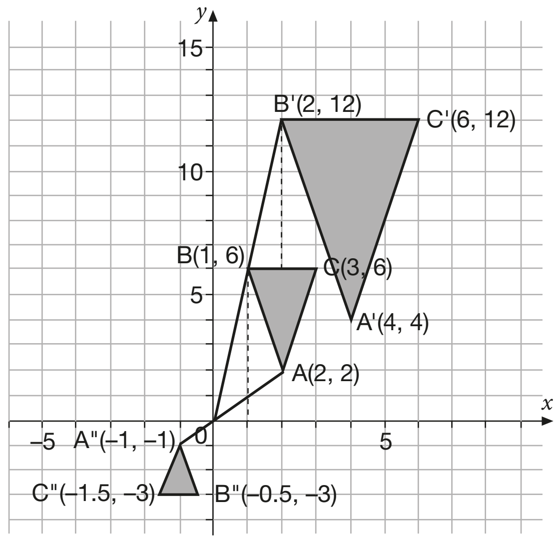
- **This rule is also true for an anti-clockwise rotation of 270° around the origin, because it will end in the same position as the figure above.**



- If the same triangle is taken again and A is joined to the origin, and AO is produced the same length on the opposite side of the origin, A is rotated 180° clockwise or anti-clockwise around the origin.

  The same process could be repeated to find B' and C':
  - **The general rule can, therefore, be deduced. It states that if any point is rotated through 180° clockwise or anti-clockwise around the origin, then $(x, y) \to (-x, -y)$.**

2. When the origin is the centre of enlargement, special rules can also be deduced.



If $\triangle$ABC is used again, OB is drawn and OB' is drawn so that it is twice the length of OB. The same process is followed to find A' and C':

- Students will see that BC = 2 units and B'C' = 4 units and, therefore, all the sides of $\triangle$A'B'C' are twice the length of the sides of $\triangle$ABC. Also, $\frac{OB'}{OB} = \frac{2\sqrt{37}}{\sqrt{37}} = 2$.
- If students look at the coordinates of corresponding points, the coordinates of the enlarged triangle are twice the coordinates of the original triangle.
  - **So, in general, if the sides of the enlarged triangle are *k* times longer than those of the original triangle, then $(x, y) \to (kx, ky)$.**

If on the other hand, the enlargement factor is negative, the image is on the other side of the origin, the centre of enlargement.

- In the figure here, $\frac{OA''}{OA} = \frac{\sqrt{2}}{\sqrt{8}} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$, but because $\triangle$A"B"C" is on the other side of the origin, the enlargement factor is $-\frac{1}{2}$.
- So, the coordinates of A" will be $-\frac{1}{2}$ times the coordinates of A, and this is also true for the coordinates of B" and C".

Chapter 16 Algebraic processes 4: Gradients of straight lines and curves

Learning objectives

By the end of this chapter, the students should be able to:

1. Calculate the gradient of a straight line.
2. Sketch the graph of a straight line of which the equation is $y = mx + c$, where m is the gradient of the line and c is the y-intercept.
3. Determine the equation of a straight line from given data.
4. Draw the tangent to a curve at a given point.
5. Use the tangent to find an approximate value for the gradient of the curve at a given point.

Teaching and learning materials

Students: Textbook, graph exercise book, ruler and pencil and mathematical sets.

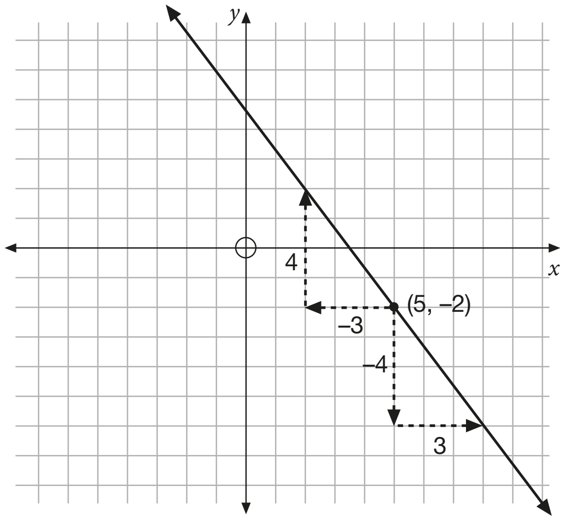
Teacher: Graph board or graph paper transparencies, a ruler, transparency pens and an overhead projector, if available, and a mirror.

Areas of difficulty

- Students find the concept of gradient difficult and often divide the change in the x-values by the change in y-values. Emphasise that the gradient $= \frac{\text{change in } y}{\text{change in } x} = \frac{\text{vertical change}}{\text{horizontal change}}$.
- The reason why the gradient of a line parallel to the y-axis (x = constant number) is not defined, is because the horizontal change = 0. This is because the x-value stays unchanged:
  - Gradient $= \frac{\text{change in } y}{\text{change in } x} = \frac{\text{change in } y}{0}$ and division by 0 is undefined.
  - You can explain as follows why division by 0 is undefined:
    - $\frac{8}{2} = 4$ = to a certain number, because $2 \times 4 = 8$. $\frac{0}{2} = 0$, because $2 \times 0 = 0$.
    - If $\frac{8}{0}$, this number $\times$ 0 must be equal to 8 and that is impossible.
    - Division by 0 is, therefore, undefined.
- The gradient of a horizontal line, y = a constant number, is equal to 0, because there is no change in the y-values. Therefore, the gradient $= \frac{\text{change in } y}{\text{change in } x} = \frac{0}{\text{change in } x} = 0$.
- When drawing the graph of a function represented by $y = ax^2 + bx + c$, students find it difficult to actually draw an accurate curve.
  - If they do not have plastic curves, let them bend a piece of soft wire in such a way that the wire passes through all the points they plotted on the graph paper.
  - Then let them trace the curve of the wire with a pencil.
- Sometimes students find it difficult to identify when a gradient is positive and when it is negative. Teach them that:
  - If the line goes up or from the left to the right like an incline on a mountain road, then the gradient is positive (+).
  - If the line goes down from the left to the right like a decline on a mountain road, then the gradient is negative (–).

Supplementary worked examples

1. Sketch the line that passes through the point (5, –2) and has a gradient of $-\frac{4}{3}$.



These steps can be followed:

a) First plot the point where you think it approximately would be.

b) For the gradient = $\frac{-4}{3}$. From the point go approximately 4 units down and 3 units to the right.

c) Or for the gradient = $\frac{4}{-3}$. From the point go approximately 3 units to the left and 4 units up.

Always measure the gradient from the given point.

- Remember for the change in the *y*-values or the vertical change:
  - Upwards is positive: ↑.
  - Downwards is negative: ↓.
- Remember for the change in *x*-values or the horizontal change:
  - To the right is positive: →.
  - To the left is negative: ←.

2. Example 5 could also be done like this:
Use the equation $y = mx + c$ of the straight line, where m represents the gradient and c represents the *y*-intercept.

Step 1: Since $m = -\frac{1}{3}$, we can write
$y = -\frac{1}{3}x + c.$

Step 2: To find c, we substitute the given point (–3, 2) into $y = -\frac{1}{3}x + c$.
$2 = -\frac{1}{3}(-3) + c$
$2 = 1 + c$
$\therefore c = 1$

Step 3: The answer is: $y = -\frac{1}{3}x + 1$.

3. Example 6 could also be done like this:
Again use the equation $y = mx + c$ of the straight line, where m represents the gradient and c represents the *y*-intercept.

Step 1: Determine the gradient: $m = \frac{6-4}{-2-1} = \frac{2}{-3}$
$= -\frac{2}{3}$ or $m = \frac{4-6}{1-(-2)} = -\frac{2}{3} = -\frac{2}{3}$
(It does not matter which *y*-value you take first, as long as you take the corresponding *x*-value also first.)

Step 2: Now the equation so far is:
$y = -\frac{2}{3} + c.$

Step 3: Substitute any of the two given points into $y = -\frac{2}{3} + c$.
If we use the point (1, 4):
$4 = -\frac{2}{3}(1) + c$
$\therefore c = 4 + \frac{2}{3} = 4\frac{2}{3}$
If we use the point (–2, 6):
$6 = -\frac{2}{3}(-2) + c = \frac{4}{3} + c = 1\left(\frac{1}{3}\right) + c$
$\therefore c = 6 - 1\frac{1}{3} = 4\frac{2}{3}$

Step 4: The answer is $y = -\frac{2}{3} + 4\frac{2}{3}$.

Chapter 17 Algebraic processes 5: Algebraic fractions

Learning objectives

By the end of this chapter, the students should be able to:

1. Use factorisation to simplify algebraic fractions by reducing them to their lowest terms.
2. Multiply and divide algebraic fractions, reducing them to their lowest terms.
3. Add and subtract algebraic fractions to give a single algebraic fraction.
4. Use substitution of numerical values or algebraic terms to simplify given algebraic fractions.
5. Solve equations that contain algebraic fractions.
6. Understand what is meant by an undefined fraction.
7. Determine the values that make a fraction undefined.

Teaching and learning materials

Students: Textbook, exercise book and writing materials.
Teacher: Textbook, chalkboard and chalk.

Areas of difficulty

- Students find it difficult to understand why $x - 3 = -(3 - x)$ or why $x + 3 = -(-x - 3)$.
  You can explain this as follows:
  - $6 - 3 = 3$ and $-(3 - 6) = -(-3) = 3$, so $x - 3 = -(3 - x)$
  - $6 + 3 = 9$ and $-(-6 - 3) = -(-9) = 9$.
- As teacher you should realise that the word simplify means that an algebraic expression is written in another form. This other form is simpler or easier to use in certain situations.
  - If you want to substitute x by -2, for example, you would choose to write $9x - 5x - 8x + 10x$ as $6x$. It is much easier to substitute x by -2 in the expression $6x$, than in the expression $9x - 5x - 8x + 10x$.
  - So this expression is simpler for the purpose of substitution. In this chapter, the numerator and denominator of algebraic fractions must be written in factor form before the fraction can be written in its lowest terms.
  - When multiplying and dividing algebraic fractions, their numerators and denominators should also be written in factor form before these fractions can be divided and multiplied.
  - For these algebraic fractions, the factor form of their numerators and denominators is the simplest form for these specific purposes.
- When writing an algebraic fraction in its lowest terms, students tend to "cancel" terms when they should actually "cancel" or divide factors.
  - You could emphasise that they first factorise the numerator and denominator of the fraction before dividing by doing a numerical example, such as the example below before you do Examples 1 and 2.
  - Although we do not write numerical fractions like this when we write them in their simplest form, this is actually how we think when we do the calculation.
  - $\frac{24}{36} = \frac{\cancel{2} \times \cancel{2} \times 2 \times \cancel{3}}{\cancel{2} \times \cancel{2} \times 3 \times \cancel{3}} = \frac{2}{3}$
- When multiplying and dividing algebraic fractions, students tend to also cancel terms instead of first factorising the numerators and denominators of fractions. Before you do Examples 4 to 6, you could also do this example to try and explain why students first must factorise:
  - $\frac{12}{35} \times \frac{28}{45} \div \frac{24}{75} = \frac{12}{35} \times \frac{28}{45} \times \frac{75}{24}$
    $= \frac{\cancel{2} \times \cancel{2} \times \cancel{3}}{\cancel{5} \times \cancel{7}} \times \frac{\cancel{2} \times 2 \times \cancel{7}}{\cancel{3} \times \cancel{3} \times \cancel{5}} \times \frac{\cancel{3} \times \cancel{5} \times \cancel{5}}{\cancel{2} \times \cancel{2} \times \cancel{2} \times 3}$
    $= \frac{2}{3}$

- When adding or subtracting algebraic fractions, students should remember that the line underneath a numerator that has more than one term, acts like a bracket:
  - $\frac{4x+3}{4x} - \frac{x-3}{12x} - \frac{3x-2}{3x}$
  - $\frac{3(4x+3)}{3 \times 4x} - \frac{1(x-3)}{12x \times 1} - \frac{4(3x-2)}{4 \times 3x}$

  (writing fractions with equal denominators, the LCM of $4x$, $3x$ and $12x$ is used)
  (the principle here is that, if one multiplies the numerator and denominator by the same number, one actually multiplies by 1 and then the fraction stays unchanged)
  - $= \frac{3(4x+3) - 1(x-3) - 4(3x-2)}{12x}$

  (brackets are used to avoid the mistake of not multiplying all the terms of the numerator)
  - $= \frac{12x + 9 - x + 3 - 12x + 8}{12x}$

    $= \frac{-x+20}{12x}$, where $x \neq 0$.
- Students tend to treat the addition and subtraction of algebraic fractions the same as when solving equations with fractions.
  - It is very important that they realise that when they add or subtract of algebraic fractions, they actually work with algebraic expressions.
  - When they solve of equations with fractions it is a completely different matter. The differences are shown below:

Algebraic fractions (algebraic expressions)	**Equations**
• Have only one side. • Are added by putting the whole expression on a common denominator, which is the LCM of all the denominators. • Ends with one fraction, for example, $\frac{-3x}{4}$.	• Have two sides (left-hand side and right-hand side). • Both sides can be multiplied by the LCM of the denominators. • Remove the fraction to make it easier to solve the equation. • Ends with a solution, for example, $x = 2$.

Adding algebraic fractions (algebraic expressions)	**Solving equations**
$\frac{3x-4}{4} - \frac{3x-2}{2}$ $= \frac{1(3x-4) - 2(3x-2)}{4}$ $= \frac{3x - 4 - 6x + 4}{4}$ $= \frac{-3x}{4}$	$\frac{3x-4}{4} - \frac{3x-2}{2} = 2$ $1(3x-4) - 2(3x-2) = 4 \times 2$ $3x - 4 - 6x + 4 = 8$ $-3x = 8$ $x = -\frac{8}{3}$

When students test whether their solution(s) of an equation is correct, they should substitute these root(s) in the original equation.
- They then tend to work with the two sides of the equation simultaneously. This can lead to mathematical errors.
- Emphasise and insist that they substitute the root of the equation in its left-hand side and work out the answer. Then they should substitute the root in the right-hand side of the equation and work out the answer.
- If the two answers are equal, they should then write: LHS = RHS, therefore, 2, for example, is the root of the equation.
- For example: The equation $\frac{x-5}{3} - \frac{9-x}{2} = \frac{x+4}{5}$ was solved and $x = 11$. To test whether this answer is correct, students should write the following:

  $$\text{LHS} = \frac{11-5}{3} - \frac{9-11}{2} = \frac{6}{3} - \frac{(-2)}{2} = 2 + 1 = 3$$
  $$\text{RHS} = \frac{x+4}{5} = \frac{11+4}{5} = \frac{15}{5} = 3$$
  $\therefore$ LHS = RHS
  $\therefore$ 11 is a root of the equation.

Supplementary worked examples

Solve for x:

1. $\frac{1}{2} + \frac{1}{3x} = \frac{2-x}{6x}$
2. $\frac{4}{x+3} + \frac{4}{4-x^2} = \frac{5x-5}{x^2+x-6}$
3. $\frac{x+9}{x^2-9} + \frac{1}{x+3} = \frac{2}{x-3}$

Solutions

1. $(6x)\frac{1}{2} + (6x)\frac{1}{3x} = (6x)\frac{2-x}{6x}$

   $3x + 2 = 2 - x$
   $4x = 0$
   $x = 0$

   $\therefore$ There is no solution because $x \neq 0$.

2.
$$\frac{4}{x+3} + \frac{4}{-(x^2-4)} = \frac{5x-5}{x^2+x-6}$$
$$\frac{4}{x+3} - \frac{4}{(x+2)(x-2)} = \frac{5x-5}{(x+3)(x-2)}$$
$$(x+3)(x-2)(x+2)\frac{4}{x+3} - (x+3)(x-2)(x+2)\frac{4}{(x+2)(x-2)} = (x+3)(x-2)(x+2)\frac{5x-5}{(x+3)(x-2)}$$
$$4(x-2)(x+2) - 4(x+3) = (5x-5)(x+2)$$
$$4x^2 - 16 - 4x - 12 = 5x^2 + 5x - 10$$
$$x^2 + 9x + 18 = 0$$
$$(x+6)(x+3) = 0$$
$\therefore x = -6$ or $x = -3$, but $x \neq -3$

$\therefore x = -6$

3.
$$\frac{x+9}{(x+3)(x-3)} + \frac{1}{x+3} = \frac{2}{x-3}$$
$$x + 9 + x - 3 = 2(x+3)$$
$$2x + 6 = 2x + 6$$
$\therefore x \in \mathbb{R}$, but $x \neq 3$ and $x \neq -3$.

Chapter 18 Numerical processes 3: Sequences and series

Learning objectives

By the end of this chapter, the students should be able to:

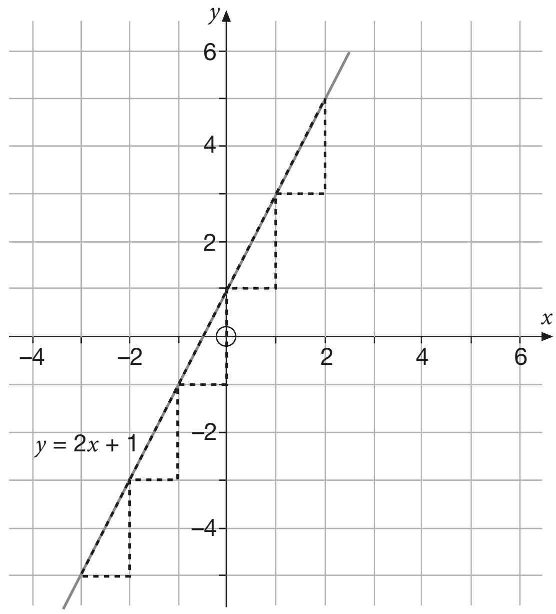
1. Determine the rule that generates a sequence of terms, extending the sequence as required.
2. Find an algebraic expression for the nth term of a simple sequence.
3. Define an arithmetic progression in terms of its common difference, d, and first term, a.
4. Find the nth term of an arithmetic progression, using the formula $U_n = a + (n - 1)d$.
5. Define a geometric progression in terms of its common ratio, r, and first term, a.
6. Find the nth term of a geometric progression, using the formula $U_n = ar^{n-1}$.
7. Distinguish between a sequence and a series.
8. Calculate the sum of the first n terms of an arithmetic series.
9. Calculate the sum of the first n terms of a geometric series.
10. Calculate the sum to infinity of a geometric series.
11. Apply sequences and series to numerical and real-life problems.

Teaching and learning materials

Students: Textbook, exercise book and writing materials.
Teacher: Textbook, chalkboard and chalk.

Areas of difficulty

- Students do not always know when to use the formula $S_n = \frac{n}{2}(a + 1)$ or when to use the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$ to determine the sum of an arithmetic series. Teach them that the first formula can only be used if they know the last term of the series. The second formula is used when the last term is not known.
- We know that the equation of a straight line can be written as $y = mx + c$, where m is the gradient and c is the y-intercept.
  - The gradient of a straight line actually means the quantity by which the y-values increase per 1 unit increase of the x-values. The gradient for a straight line is constant.
  - So this means that we actually have the same condition here as with an arithmetic sequence, where there is a constant increase (or decrease) from one term to the next term.
  - If the gradient is equal to 2, the y-values constantly increase by 2 when the x-values increase by 1. This diagram shows this:



The equation of the graph is $y = 2x + 1$. If we show some of the integer points visible in the diagram in a table, we see the following:

x	–3	–2	–1	0	1	2
y	–5	–3	–1	1	3	5

If we just take the y-values –5, –3, –1, 1, 3, 5, …; we see that this actually is an arithmetic sequence with a constant difference of 2.

- So, the nth term can also be written as $U_n = dn + c$, where d is the constant difference and c is the value we must add to obtain the first term.
- The nth term of an arithmetic sequence can be determined by using this method:
  1. Write down the common difference, d.
  2. Write $U_n = dn + c$.
  3. Now substitute the first term and $n = 1$. So Example 3(c) on page 204 can also be done like this:

$$d = 3$$
$$U_n = 3n + c$$
$$9 = 3(1) + c$$
$$\therefore c = 6$$
$$\therefore U_n = 3n + 6$$

- The method used in Example 8 on page 207 was first used by Friedrich Gauss (1777–1855), a well-known German mathematician. When he was still in school, his teacher one day tried to keep the class busy by telling them to add all the natural numbers from 1 to 100. Gauss had the answer within seconds by using this method.
- We can only get the sum of an infinite geometric series if we can see what the answer will tend to become if we take more and more terms. We can deduce the sum of an infinite geometric series as follows:

1. Consider this infinite series: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, where $r = \frac{1}{2}$.

   a) $S_1 = \frac{1}{2}$

   $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

   $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

   $S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

   $S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$

   $S_6 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{63}{64}$

   b) What does the answer tend to become?

2. Consider this infinite series: $\frac{1}{2} + (-\frac{1}{4}) + \frac{1}{8} + \ldots$, where $r = -\frac{1}{2}$.

   a) $S_1 = \frac{1}{2}$

   $S_2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

   $S_3 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

   $S_4 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} = \frac{5}{16} = 0.3125$

   $S_5 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} = \frac{11}{32} = 0.34375$

   $S_6 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} = \frac{21}{64}$
   $= 0.328125$

   $S_7 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \frac{1}{128} = \frac{43}{128}$
   $= 0.3359375$

   b) What does the answer tend to become?

3. Complete the following for the infinite series in number 1:

   $S_n = \frac{a(1 - r^n)}{1 - r}$ $(r < 1)$ (The symbol ∞ means "infinite")

   $S_\infty = \frac{\frac{1}{2}(1 - (\frac{1}{2})^\infty)}{1 - \frac{1}{2}}$ $(r < 1)$

   $= \frac{\frac{1}{2} - \frac{1}{2}(\frac{1}{2})^\infty}{\frac{1}{2}} = \frac{\cdots}{\cdots} - \frac{\cdots}{\cdots}$

   $= 1 - \frac{\cdots}{\cdots}$

   $= 1$ (because $\frac{\cdots}{\cdots}$ to a very large power becomes so small that we can leave it out)

4. Complete the following for the infinite series in number 2:

   $S_n = \frac{a(1 - r^n)}{1 - r}$ $(r > -1)$

   $S_\infty = \frac{\frac{1}{2}(1 - (-\frac{1}{2})^\infty)}{1 - (-\frac{1}{2})}$ $(r > -1)$

   $= \frac{\frac{1}{2} - \frac{1}{2}(-\frac{1}{2})^\infty}{\frac{3}{2}}$

   $= \frac{\cdots}{\cdots} - \frac{\cdots}{\cdots}$

   $= \frac{1}{3} - \frac{\cdots}{\cdots}$

   $= \frac{1}{3}$ (because $\frac{\cdots}{\cdots}$ to a very big power becomes so small that we can leave it out)

Solutions

1. b) 1
2. b) $\frac{1}{3}$
3. $\frac{\frac{1}{2}}{\frac{1}{2}} - \frac{\frac{1}{2}(\frac{1}{2})^\infty}{\frac{1}{2}} = 1 - (\frac{1}{2})^\infty = 1$ (because $(\frac{1}{2})$ to a very big power becomes so small that we can leave it out)

4. $\frac{\frac{1}{2}}{\frac{3}{2}} - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)^{\infty}}{\frac{3}{2}} = \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^{\infty} = \frac{1}{3}$

(because $\left(-\frac{1}{2}\right)$ to a very big power becomes so small that we can leave it out)

- From what was just done, we noticed that we could never add all the terms of an infinite series. We can only say what the answer will tend to become, if we add more and more terms.
  - If r is a fraction between -1 and 1, then when we take that fraction to a very big power; the answer becomes so small that we can ignore that part. We say that the series converges to a certain value.
  - If r is not a fraction between -1 and 1, then the series cannot converge to a certain value because then there is not a fraction that would become so small that we can ignore that part.
  - The sum of a series like that will just grow infinitely big.

  So, for an infinite series the following is true:
  - $S_\infty = \frac{a(1 - r^\infty)}{1 - r}$ $(-1 < r < 1)$

    $= \frac{a - ar^\infty}{1 - r} = \frac{a}{1 - r} - \frac{ar^\infty}{1 - r}$

    $= \frac{a}{1 - r}$ (because a fraction to a very high power becomes negligently small).
  - We, therefore, say that the infinite series **converge** to a certain value and this can only happen when $-1 < r < 1$ (that is, r is a positive or negative fraction). Then, $S_\infty = \frac{a}{1 - r}$.

Additional supplementary worked examples

1. The sum to infinity of a GS is equal to $\frac{8}{7}$ and the constant ratio is $-\frac{3}{4}$. Determine the series.
2. The sum of the first three terms of an infinite GS is equal to $\frac{129}{32}$ and $S_\infty = \frac{16}{3}$. Determine the series.
3. Calculate the values of x for which each of these infinite GS will converge:
   $\frac{-3x + 2}{2} + \left(\frac{-3x + 2}{2}\right)^2 + \left(\frac{-3x + 2}{2}\right)^3 + \ldots$
4. A tree is planted when it has a height of 2 m. During the first year, it grows to a height of 3 m and its height, therefore, increased by 1 m. Thereafter, the increase in height every year is $\frac{4}{5}$ of the growth of the previous year. What is the maximum height the tree can reach?
5. Use the sum of infinite series to write $0.\dot{1}\dot{2}$ as common fraction.

Solutions

1. $\frac{a}{1 - r} = \frac{8}{7}$

   $\frac{a}{1 - \left(-\frac{3}{4}\right)} = \frac{8}{7}$

   $a \div \frac{7}{4} = \frac{8}{7}$

   $a \times \frac{4}{7} = \frac{8}{7}$

   $4a = 8$

   $\therefore a = 2$

   So, the series is:

   $2 + 2\left(-\frac{3}{4}\right) + 2\left(-\frac{3}{4}\right)^2 + \ldots = 2 - \frac{3}{2} + \frac{9}{8} - \ldots .$

2. $\frac{a}{1 - r} = \frac{16}{3}$

   $\therefore a = \frac{16}{3}(1 - r) \ldots$ ❶

   $S_3 = \frac{a(1 - r^3)}{1 - r} = \frac{129}{32} \ldots$ ❷

   Substitute ❶ into ❷:

   $\frac{16}{3}(1 - r)\left(\frac{1 - r^3}{1 - r}\right) = \frac{129}{32}$

   $1 - r^3 = \frac{129}{32} \times \frac{3}{16}$

   $r^3 = 1 - \frac{387}{512} = \frac{125}{512} = \frac{5^3}{2^9}$

   $\therefore r = \frac{5}{8}$

   $a = \frac{16}{3}\left(1 - \frac{5}{8}\right) = \frac{16}{3}\left(\frac{3}{8}\right) = 2$

   Thus, the series is:

   $2 + 2\left(\frac{5}{8}\right) + 2\left(\frac{5}{8}\right)^2 + 2\left(\frac{5}{8}\right)^3 + \ldots = 2 + \frac{5}{4} + \frac{25}{32} + \frac{125}{256} + \ldots$

3. $-1 < \frac{-3x + 2}{2} < 1$

   $-2 < -3x + 2 < 2$

   $-4 < -3x < 0$

   $0 < x < \frac{4}{3}$

4. $1 + 1\left(\frac{4}{5}\right) + 1\left(\frac{4}{5}\right)^2 + 1\left(\frac{4}{5}\right)^3 + \ldots$

   $a = 1$ and $r = \frac{4}{5}$

   Therefore, the tree would continue to grow indefinitely.

   $S_\infty = \frac{a}{1 - r} = \frac{1}{1 - \frac{4}{5}} = 1 \div \frac{1}{5} = 5$

   However, the tree will never grow taller than 7 m.

5. $0.\dot{1}\dot{2} = 0.1212121212\ldots = 0.12 + 0.0012 + 0.000012 + \ldots$

   $a = 0.12$ and $r = 0.01$

   $S_\infty = \frac{0.12}{1 - 0.01} = \frac{12}{100} \div \frac{99}{100} = \frac{12}{100} \times \frac{100}{99} = \frac{4}{33}$

Chapter 19 Statistics 4: Measures of dispersion

Learning objectives

By the end of this chapter, the students should be able to:

1. Describe and interpret (numerically and verbally) the dispersion or spread of values in a data set.
2. Calculate the range, variance and standard deviation of a set of ungrouped values.
3. Use and interpret standard deviation in real-life situations.

Teaching and learning materials

Students: Writing materials, exercise book, Textbook, calculator or square root tables (provided on pages 246 to 247 of the Student's Book).

Teacher: Newspaper articles, reports and data sets from recent health, market, population and similar official and unofficial studies.

Areas of difficulty

- There usually are no areas of difficulty in this chapter, if the work is carefully explained.
- The work could also be made much easier if calculators are available.
- If there is a computer available, you can show the students how to do this work in Microsoft® Excel.

Supplementary worked examples

1. A chess team (which consists of 10 players) each scored these points in a particular year:
   23, 34, 39, 40, 42, 53, 56, 62, 68 and 76
   a) Calculate the average for the team.
   b) Complete this table. Then calculate the variance and standard deviation for the team.

Points (x)	$(x - \bar{x})$	$(x - \bar{x})^2$
23	$23 - 49.3 = -26.3$	691.69
34		
39		
40		
42		
53		
56		
62		
68		
76		
Total:	$\Sigma(x - \bar{x}) =$	$\Sigma(x - \bar{x})^2 = 2\ 454.1$
Mean ($\bar{x}$)		Variance $= \frac{\Sigma(x - \bar{x})^2}{n} =$
		Std deviation $= \sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}} =$

Where:
- Σ means "the sum of"
- n represents the number of chess players
- σ is the symbol for standard deviation.

c) Determine the number of players within one standard deviation from the average.

2. Look at this table that shows the number of students who got a mark out of 30 for a class test.

Marks (x)	18	19	20	21	22	23	24	25	26	27
Number of students	2	2	3	4	6	9	12	6	5	3

Copy and complete this table:

Marks (x)	**Frequency (f)**	$f \times x$	$(x - \bar{x})$	$(x - \bar{x})^2$	$(f) \cdot (x - \bar{x})^2$
18	2	36	–5.2	27.04	54.08
19	2				
20	3				
21	4				
22	6				
23	9				
24	12				
25	6				
26	5				
27	3				
Total					
Mean ($\bar{x}$)				Variance =	
				Std deviation =	

Solutions

1. a) and b)

Points (x)	$(x - \bar{x})$	$(x - \bar{x})^2$
23	23 – 49.3 = –26.3	691.69
34	–15.3	234.09
39	–10.3	106.09
40	–9.3	86.49
42	–7.3	53.29
53	3.7	13.69
56	6.7	44.89
62	12.7	161.29
68	18.7	349.69
76	26.7	712.89
Total	$\Sigma(x - \bar{x}) = 493$	$\Sigma(x - \bar{x})^2 = 2\,454.1$
Mean ($\bar{x}$)	49.3	Variance = $\frac{\Sigma(x - \bar{x})^2}{n} = 245.41$
		Std deviation = $\sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}} = 15.66556734$

In Column 1:
- *To get the total*, click in the appropriate cell and type = **sum(** . Then you select the numbers in Column 1 from **23** to **76** and type **)** . Press **enter**.
- *To get the mean*, go to the appropriate cell and press = . Then click on the total, type **/10** and press **enter**.
- In Column 2, you must subtract the mean *x*.
- Put the cursor in the first cell of Column 2 and type =, then click on **23** in Column 1 and type **–49.3** and press **enter**.
- Then move the cursor until you see a **black plus (+) sign** in the lower right-hand corner of the first cell of Column 1.
- Now move the mouse down as far as the cell next to the 76 in Column 1.

In Column 3:
- Put the cursor in the first cell of Column 3 and type =, click on **–26.3** in Column 2 and type **^2** . Press **enter**.
- Then move the cursor until you see a **black plus (+) sign** in the lower right-hand corner of the first cell of Column 3.
- Now move the mouse down as far as the cell next to the 76 in Column 1.

To get the total, you go to the appropriate cell and type = **sum(** . Then you select the numbers in Column 3 from **691,69** to **712.8** and type **)** . Press **enter**.
To get the variance, you go to the appropriate cell and type =, then click on the total of the previous cell, type **/10** and press **enter**.
To get the standard deviation, you type = . Then click on the variance in the previous cell and type **^0,5** . Press **enter**.

c) One standard deviation is approximately 16 points.
One standard deviation from the mean = 49.3 + 16 = 65.3 or 49.3 – 16 = 33.3.
You can show this by using a sketch:

33.3 65.3

The marks are 34, 39, 40, 42, 53, 56, and 62. Therefore, there are 7 players.

2. If you follow the same directions set out for Question 1 above, you can also complete this table using Microsoft® Excel.

Marks (x)	**Frequency (f)**	$f \times x$	$(x - \bar{x})$	$(x - \bar{x})^2$	$(f)\cdot(x - \bar{x})^2$
18	2	36	–5.192	26.960	53.920
19	2	38	–4.192	17.575	35.151
20	3	60	–3.192	10.191	30.572
21	4	84	–2.192	4.806	19.225
22	6	132	–1.192	1.422	8.530
23	9	207	–0.192	0.037	0.333
24	12	288	0.08	0.652	7.828
25	6	150	1.08	3.268	19.607
26	5	130	2.08	7.883	39.416
27	3	81	3.08	14.499	43.496
Total	52	1 206	0.0	87.40	
Mean ($\bar{x}$)		23.192		Variance = 5.06	
				Std deviation = 2.25	

Note: Use * for multiplication.

Chapter 20 Logical reasoning: Valid argument

Learning objectives

By the end of this chapter, the students should be able to:

1. Define and use simple and compound statements and apply them to arguments.
2. Recognise and use the symbols for negation, conjunction, disjunction, implication and equivalence.
3. Use Venn diagrams and truth tables to demonstrate connections between statements and arguments.
4. Determine whether an argument is valid or fallacious, using the chain rule where appropriate.

Teaching and learning materials

Students: Textbook and writing materials.
Teacher: Textbook, chalkboard and chalk.

Areas of difficulty

- This whole chapter has the potential to be difficult for the students. They may tend to do the work by rote by just following the examples.
- Let them work in groups and encourage them to argue about the problems.