Chapter 11 Calculus 2: Integration

Learning objectives

By the end of this chapter, the students should be able to:

1. Determine the equation of a line, or curve given its gradient function.
2. Use the inverse of differentiation to find the general solutions of simple differential equations.
3. Apply the rules of integration to integrate algebraic expressions.
4. Calculate the arbitrary constant, given sufficient relevant information.
5. Apply integral calculus to solve problems relating to:
   - coordinate geometry.
   - displacement, velocity, acceleration.
6. Evaluate definite integrals in the form $\int_a^b f(x)dx$.
7. Find the area under a curve by integrating between limits.

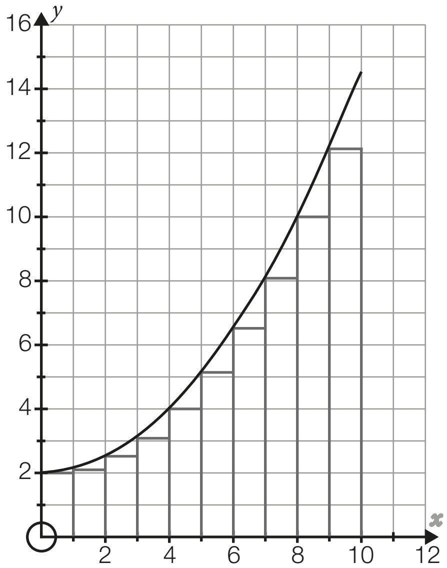
Teaching and learning materials

Students: Textbook, exercise book and writing materials.
Teacher: Charts showing standard integrals; calculus-related computer instructional materials where available.

Teaching notes

- Integration can be seen as:
  - The inverse process of differentiation, and it is, therefore, also called anti-differentiation. It means finding the function that was differentiated to find the present function. This process is also called determining the indefinite integral.
  - The mathematical process by which the exact area under a curve, bounded by the x-axis, and for a certain interval is found. This process is called determining the definite integral.
- **The rules for differentiation are:**
  1. **Integral of a constant:** $\int kdx = kx + C$, where k and C are constants.
     In words: The integral of a constant is that constant times x, and then add C, the arbitrary constant of integration.
  2. **The integral of a power of a variable, for example x^n:** $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$.
     In words: For a power of a variable (say x) add 1 to the exponent of x and divide the power of x by that same number and then add C, the arbitrary constant of integration.
- If you want to find the integral of an expression like $(2x + 3)^2$, you must first expand it by determining the square of the expression. Then, you can determine the integral of each term separately.
- You can check your answers by differentiating them to see if you get the function you had to integrate.
- The velocity is the derivative of the function that represents displacement. So, to determine the function that represents displacement, the velocity function has to be integrated.
- Acceleration is the derivative of the function that represents velocity. So, to determine the function that represents velocity, the acceleration function has to be integrated.

- To determine the definite integral, we follow these steps:
  Step 1: Use anti-differentiation to integrate the given function. (The integration constant, C, is not necessary.)
  Step 2: Substitute the biggest x-value into the integral.
  Step 3: Substitute the smallest x-value into the integral.
  Step 4: Subtract the second answer from the first answer.



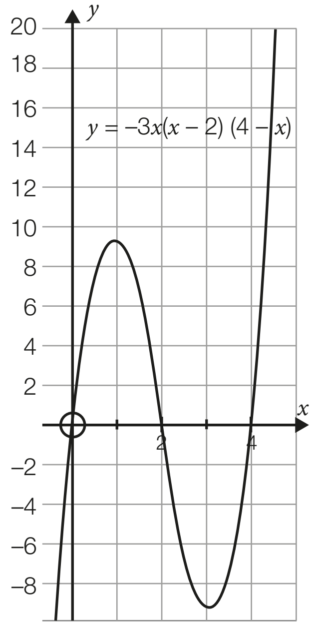
- $\int_a^b f(x)dx$ is the general notation for the precise area under the curve. This is also called the definite integral with respect to x.
  - $f(x)$ represents the heights of the rectangles for different x-values of the rectangles.
  - dx represents the breadth of each rectangle.
  - The elongated S, $\int$, represents the sum of all the infinitely thin rectangles ($dx \to 0$). We also let the number of rectangles (n) tend to ∞ ($n \to \infty$).
  - The 'a' and 'b' represent the interval $[a, b]$ for the x-values.

Areas of difficulty and common mistakes

There are some difficult concepts to understand in this chapter.
They are, for example, the fact that rectangles can be infinitely thin and that we add an infinite number of these rectangles to find the area between a curve, the x-axis and from a certain x-value to a bigger x-value.
You will have to explain this very carefully.

Supplementary worked examples

Find the exact area under this curve (between the given x-values and the x-axis) by using the definite integral:



a) $x \in [0, 2]$
b) $x \in [2, 4]$
c) $x \in [0, 4]$

Solution

a) $\int_0^2 (3x^3 - 18x^2 + 24x)dx$ (First remove the brackets)

$= \left(3\frac{x^4}{4} - 18\frac{x^3}{3} + 24\frac{x^2}{2}\right)\Big|_0^2$

$= \left(3\frac{x^4}{4} - 6x^3 + 12x^2\right)\Big|_0^2$

$= \left(\frac{3}{4}(2)^4 - 6(2)^3 + 12(2)^2\right) - 0$

$= 12 \text{ units}^2$

b) $\int_2^4 (3x^3 - 18x^2 + 24x)dx$

$= \left(3\frac{x^4}{4} - 6x^3 + 12x^2\right)\Big|_2^4$

$= \left(\frac{3}{4}(4)^4 - 6(4)^3 + 12(4)^2\right) - 12$

$= -144 - 12$

$= -156 \text{ units}^2$

$\therefore$ Area $= 156 \text{ units}^2$ (Area cannot be negative)

c) Total area $= (12 + 156) \text{ units}^2 = 178 \text{ units}^2$