Chapter 9 Coordinate geometry of straight lines

Learning objectives

By the end of this chapter, the students should be able to:

1. Recall and use various forms of the general equation of a straight line, including $y = mx + c$ and $px + qy + r = 0$.
2. Recall and use various forms of the equations of lines that are parallel to the Cartesian axes.
3. Recall and use various expressions for the gradient of a straight line.
4. Find the intercepts that a line makes with the x- and y-axis and use them to sketch the line.
5. Find the distance between two points on a Cartesian plane.
6. Find the coordinates of the midpoint of a straight line joining two points of the Cartesian plane.
7. Recall and use the conditions for two lines to be a) parallel and b) perpendicular.
8. Calculate the angle between two lines on the Cartesian plane.

Teaching and learning materials

Students: Textbook, exercise book, writing materials, graph paper, mathematical instruments, tangent tables (see page 276 of the textbook) or scientific calculator.
Teacher: Graph board, chalkboard instruments, tangent tables or calculator. If an overhead projector is available: graph transparencies, and projector pens.

Glossary of terms

Cartesian plane is a flat surface in which the position of each point is defined by its distance along a horizontal axis (the x-axis) and its distance along a vertical axis (the y-axis) in this order.

- These axes intersect at the point (0, 0) called the origin.
- The values on the x-axis to the left of the origin are negative and to the right they are positive.
- Above the origin the values on the y-axis are positive and below the origin they are negative.
- On this plane, each point can be found by its coordinates (x, y).
- This plane is named after the mathematician René Descartes (1596–1650).

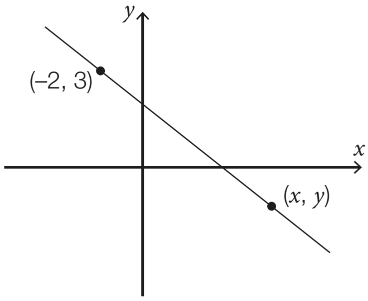
Teaching notes

- Explain to the students that the gradient of a line parallel to the x-axis ($y = p$) is 0.
  - This is, because if we write the equation in the form $y = mx + c$, where m represents the gradient, it would be $y = 0 \times x + p$.
  - We can also say that in $m = \frac{y_2 - y_1}{x_2 - x_1}$, the y-values do not change and $y_2 - y_1$ is, therefore, equal to 0.
  - So, $m = \frac{0}{x_2 - x_1} = 0$.

  The gradient of a line parallel to the y-axis, $x = q$, is not defined.
  In $m = \frac{y_2 - y_1}{x_2 - x_1}$ the x-values do not change and $x_2 - x_1$ is, therefore, equal to 0.
  So, $m = \frac{y_2 - y_1}{0}$, which is undefined.
  Remember that division by 0 is undefined because we say that, for example, $\frac{10}{5} = 2$ because $2 \times 5 = 10$.
  If, however, we say $\frac{10}{0}$ = a number, that number × by 0 must be equal to 10, which is impossible.
  So, division by 0 is undefined.
- Teach the students that one finds where the graph intersects the x-axis (x-intercepts) by making $y = 0$ in the equation of the graph.
  The reason we do this, is because on the x-axis all the y-coordinates are equal to 0.
  One finds where the graph intersects the y-axis (y-intercept) by making $x = 0$ in the equation of the graph.

The reason we do this, is because on the y-axis all the x-coordinates are equal to 0.

- There are two methods to find the equation of a straight-line graph, if the gradient and another point are given.
  Let us say we want to find the equation of a line that passes through the point $(-2, 3)$ and has a gradient of -3.
  Method 1:



You can say "if there is any other point on the graph (x, y), then the gradient is $\frac{y-3}{x-(-2)} = -3$".

So, $y - 3 = -3(x + 2)$

$y = -3x - 6 + 3$

$= -3x - 3$

Method 2:

Write the equation of the line from the information you have: $y = -3x + \text{c}$.

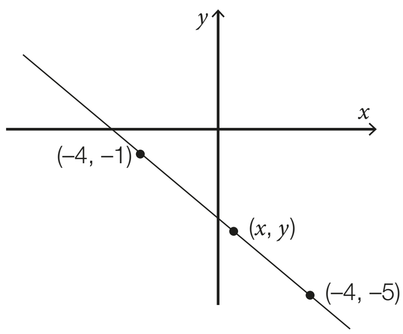
Now, substitute the point that was given:

$3 = -3\,(-2) + c$

$c = 3 - 6 = -3$

$\therefore y = -3x - 3$

- There are also two methods to find the equation of a line if two points on the line are given:
  Let us say we want to find the equation of the straight-line graph that passes through the points $(-4, -1)$ and $(4, -5)$.



Method 1:

If there is any point (x, y) on the line the gradient between this point and $(-4, -1)$ and between the points $(-4, -1)$ and $(4, -5)$ are equal.

So, $\frac{-1-y}{-4-x} = \frac{-1-(-5)}{-4-4}$

$= \frac{-1+5}{-8}$

$= \frac{4}{-8}$

$= -\frac{1}{2}$

$\frac{-y-1}{-4-x} = \frac{-1}{2}$

$2(-y-1) = -1(-4-x)$

$-2y - 2 = x + 4$

$-2y = x + 6$

$y = -\frac{1}{2}x - 3$

Method 2:

First work out the gradient:

$m = \frac{-1-(-5)}{-4-4} = \frac{-1+5}{-8}$

$= \frac{4}{-8}$

$= -\frac{1}{2}$

Now, you can write $y = -\frac{1}{2}x + c$.

Substitute any one of the two points, for example, $(-4, -1)$:

$-1 = -\frac{1}{2}(-4) + c$

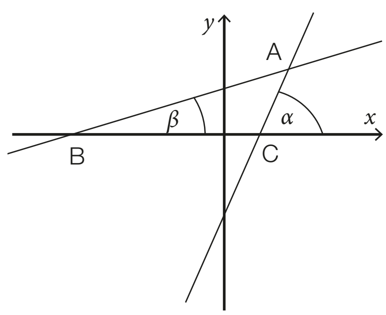
$-1 = 2 + c$

$\therefore c = -3$

$\therefore y = -\frac{1}{2}x - 3$

Explain to the students that they can choose to use any one of the two methods.

- Explain to the students that when we work out the midpoint of a line, we actually find the average between the two x-coordinates and the average between the two y-coordinates.
- When you want to calculate the angle between two lines, an alternate way to do this is to use the fact that the exterior angle of a triangle is equal to the sum of the two opposite interior angles.
  So, Example 6 could also be done like this:

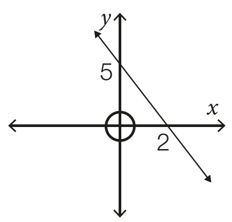


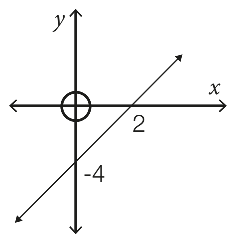
Then $\angle BAC = \alpha - \beta$, because $\alpha = \beta + \angle BAC$.

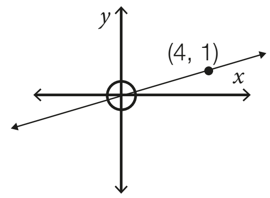
Areas of difficulty and common mistakes

- For some reason students tend to get the gradient wrong as $\frac{x_2 - x_1}{y_2 - y_1}$. Tell them to check their work and to remember how the gradient is obtained.
- Sometimes students forget how to find the midpoint of a line.
  Tell them that, if we want to find the x-coordinate in the middle of two other x-coordinates, then we always add the two x-coordinates and divide the answer by 2.
  The same principle applies for finding the y-coordinate of the midpoint of a line segment.
- Students find it difficult to find the angle between two lines.
  - Tell them to always first make a sketch of the two lines to find out what they can do to work out the angle between the two lines by using the fact that their gradients are the tan ratios of the angles the lines make with the x-axis.
  - The angles the lines make with the x-axis are always measured anti-clockwise from the x-axis.
  - These angles are called the angles of inclination.
- Students sometimes also use the square root sign inappropriately when they work out the length of a line. For example:
  AB = $\sqrt{64 + 36} = 8 + 6 = 14$, instead of $\sqrt{100} = 10$.
  Or, they write $AB^2 = 64 + 36 = 100 = 10$ instead of writing $AB^2 = 64 + 36 = 100$.
  $\therefore AB = 10$

Supplementary worked examples

1. Determine the values of a and b when C is the midpoint of AB, if:
   a) A(a, 4), B(3, b) and C(4, −6)
   b) A(a, −7), B($2a$, b) and C(3, 4b)
2. For each of these write down the gradient of the line that is i) parallel to, and ii) perpendicular to, the given line:
   a) $y = 3x + 2$
   b) $3x + 2y = 6$
   c)


   d)


   e)


3. Draw sketch graphs of:
   a) $y = -2x$
   b) $y = -2$
   c) $x = 2$
   d) $y = \frac{1}{2}x$

Solution

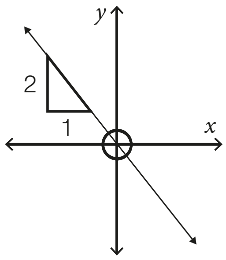
1. a)
$$\frac{a+3}{2} = 4$$
$$\therefore a + 3 = 8$$
$$\therefore a = 5$$
$$\frac{4+b}{2} = -6$$
$$\therefore 4 + b = -12$$
$$\therefore b = -16$$
   b)
$$\frac{a+2a}{2} = 3$$
$$3a = 6$$
$$\therefore a = 2$$
$$\frac{-7+b}{2} = 4b$$
$$-7 + b = 8b$$
$$-7b = 7$$
$$\therefore b = -1$$
2. a) i) 3 ii) $-\frac{1}{3}$
   b) First write the equation in gradient y-intercept form:
   $2y = -3x + 6$ (add $-3x$ to both sides)
   $y = -\frac{3}{2}x + 3$ (divide both sides by 2)
   i) $-\frac{3}{2}$
   ii) $-\frac{3}{2} \times \frac{2}{3} = -1$, so the answer is $\frac{2}{3}$.
   c) The gradient is negative and equal to $-\frac{5}{2}$.
   i) $-\frac{5}{2}$
   ii) $-\frac{5}{2} \times \frac{2}{5} = -1$, so the answer is $\frac{2}{5}$.
   d) The gradient is positive and equal to 2.
   i) 2
   ii) $2 \times \left(-\frac{1}{2}\right) = -1$, so the answer is $-\frac{1}{2}$.

e) The gradient is positive and equal to $\frac{1}{4}$.

i) $\frac{1}{4}$

ii) $\frac{1}{4} \times (-4) = -1$, so the answer is -4.

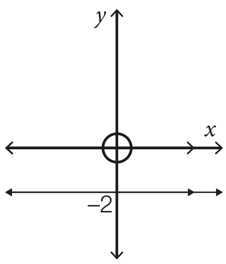
3. a)



When the y-intercept = 0, we cannot use the intercepts with the axes to draw the graph, because the x- and y-intercepts are equal and we need at least two points to draw the graph. So, we show the gradient and the fact that the line passes through the origin.

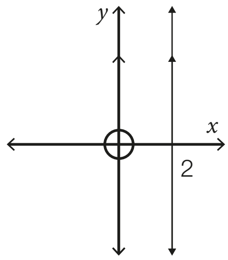
Note the direction of the line because of the fact that the gradient is negative.

b)



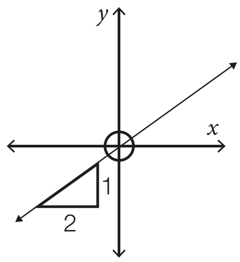
You must show that the graph is parallel to the x-axis, because it is a sketch and facts are only facts if they are shown on a sketch

c)



You must show that the graph is parallel to the y-axis.

d)



When the y-intercept = 0, we cannot use the intercepts with the axes to draw the graph, because the x- and y-intercepts are equal and we need at least two points to draw the graphs. So, we show the gradient and the fact that the line passes through the origin.

Note the direction of the line because of the fact that the gradient is positive.