New General Mathematics
FOR SENIOR SECONDARY SCHOOLS

TEACHER’S GUIDE

PEARSON
New General Mathematics
for Secondary Senior Schools 3

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Contents

Review of SB1 and SB2

Chapter 1: Numerical processes 1: Surds
Chapter 2: Numerical processes 2: Theory of logarithms
Chapter 3: Algebraic processes 1: Linear and quadratic equations
Chapter 4: Mensuration 1: Length, area and volume
Chapter 5: Numerical processes 3: Commercial arithmetic
Chapter 6: Trigonometry: Angles between 0° and 360°
Chapter 7: Mensuration: Latitude and longitude
Chapter 8: Numerical processes 4: Matrices
Chapter 9: Coordinate geometry of straight lines
Chapter 10: Calculus 1: Differentiation
Chapter 11: Calculus 2: Integration
Chapter 12: Statistics: Mean and standard deviation
Chapter 13: Revision of numerical processes
Chapter 14: Revision of indices, logarithms and surds
Chapter 15: Revision of algebraic processes and calculus
Chapter 16: Equations and inequalities
Chapter 17: Algebraic graphs, relations and functions
Chapter 18: Statistics and probability
Chapter 19: Geometry and vectors
Chapter 20: Mensuration
Chapter 21: Trigonometry
Chapter 22: Constructions and loci
1. Learning objectives
1. Number and numeration
2. Algebraic processes
3. Geometry and mensuration
4. Statistics and probability

2. Teaching and learning materials
Teachers should have the Mathematics textbook of the Junior Secondary School Course, and Book 1 and Book 2 of the Senior Secondary School Course.

Students should have:
1. Book 1 and Book 2
2. An Exercise book
3. Graph paper
4. A scientific calculator, if possible.

3. Glossary of terms

**Algebraic expression** A mathematical phrase that can contain ordinary numbers, variables (such as \(x\) or \(y\)) and operators (such as add, subtract, multiply, and divide). For example, \(3x^2y – 3y^2 + 4\).

**Angle** A measure of rotation or turning and we use a protractor to measure the size of an angle.

**Angle of depression** The angle through which the eyes must look downward from the horizontal to see a point below.

**Angle of elevation** The angle through which the eyes must look upward from the horizontal to see a point above.

**Balance method** The method by which we add, subtract, multiply or divide by the same number on both sides of the equation to keep the two sides of the equation equal to each other or to keep the two sides balanced. We use this method to make the two sides of the equation simpler and simpler until we can easily see the solution of the equation.

**Cartesian plane** A coordinate system that specifies each point in a plane uniquely by a pair of numerical coordinates, which are the perpendicular distances of the point from two fixed perpendicular directed lines or axes, measured in the same unit of length. The word Cartesian comes from the inventor of this plane namely René Descartes, a French mathematician.

**Coefficient** a numerical or constant or quantity \(\neq 0\) placed before and multiplying the variable in an algebraic expression (for example, 4 in \(4x^2\)).

**Common fraction (also called a vulgar fraction or simple fraction)** Any number written as \(\frac{a}{b}\) where \(a\) and \(b\) are both whole numbers and where \(a < b\).

**Coordinates** of point A, for example, (1, 2) gives its position on a Cartesian plane. The first coordinate (\(x\)-coordinate) always gives the distance along the \(x\)-axis and the second coordinate (\(y\)-coordinate) gives the distance along the \(y\)-axis.

**Data** Distinct pieces of information that can exist in a variety of forms, such as numbers. Strictly speaking, data is the plural of datum, a single piece of information. In practice, however, people use data as both the singular and plural form of the word.

**Decimal place values** A positional system of notation in which the position of a number with respect to the decimal point determines its value. In the decimal (base 10) system, the value of each digit is based on the number 10. Each position in a decimal number has a value that is a power of 10.

**Denominator** The part of the fraction that is written below the line. The 4 in \(\frac{3}{4}\), for example, is the denominator of the fraction. It also tells you what kind of fraction it is. In this case, the kind of fraction is quarters.

**Direct proportion** The relationship between quantities whose ratio remains constant. If \(a\) and \(b\) are directly proportional, then \(\frac{a}{b} = a\) constant value (for example, \(k\)).

**Direct variation** Two quantities \(a\) and \(b\) vary directly if, when \(a\) changes, then \(b\) changes in the same ratio. That means that:
* If \(a\) doubles in value, \(b\) will also double in value.
* If \(a\) increases by a factor of 3, then \(b\) will also increase by a factor of 3.

**Directed numbers** Positive and negative numbers are called directed numbers and are shown on a number line. These numbers have a certain direction with respect to zero.
* If a number is positive, it is on the right-hand side of 0 on the number line.
* If a number is negative, it is on the left-hand side of the 0 on the number line.
Edge  A line segment that joins two vertices of a solid.

Elimination  the process of solving a system of simultaneous equations by using various techniques to successively remove the variables.

Equivalent fractions  Fractions that are multiples of each other, for example, \( \frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{3 \times 3}{4 \times 3} \ldots = \) and so on.

Expansion  of an algebraic expression means that brackets are removed by multiplication.

Faces of a solid  A flat (planar) surface that forms part of the boundary of the solid object; a three-dimensional solid bounded exclusively by flat faces is a polyhedron.

Factorisation of an algebraic expression  means that we write an algebraic expression as the product of its factors.

Graphical method used  to solve simultaneous linear equations means that the graphs of the equations are drawn. The solution is where the two graphs intersect (cut) each other.

Highest Common Factor (HCF)  of a set of numbers is the highest factor that all those numbers have in common or the highest number that can divide into all the numbers in the set.

The HCF of 18, 24 and 30, for example, is 6.

Inverse proportion  The relationship between two variables in which their product is a constant.

When one variable increases, the other decreases in proportion so that the product is unchanged.

If \( b \) is inversely proportional to \( a \), the equation is in the form \( b = \frac{k}{a} \) (where \( k \) is a constant).

Inverse variation:  Two quantities \( a \) and \( b \) vary inversely if, when \( a \) changes, then \( b \) changes by the same ratio inversely. That means that:

* If \( a \) doubles, then \( b \) halves in value.
* If \( a \) increases by a factor of 3, then \( b \) decreases by a factor of \( \frac{1}{3} \).

Joint variation  of three quantities \( x \), \( y \) and \( z \) means that \( x \) and \( y \) are directly proportional, for example, and \( x \) and \( z \) are inversely proportional, for example. So \( x \propto \frac{y}{z} \) or \( x = \frac{kz}{y} \), where \( k \) is a constant.

Like terms  contain identical letter symbols with the same exponents. For example, \(-3x^2y^3\) and \(5x^2y^3\) are like terms but \(3x^2y^3\) and \(3xy\) are not like terms. They are unlike terms.

Lowest Common Multiple (LCM)  of a set of numbers is the smallest multiple that a set of numbers have in common or the smallest number into which all the numbers of the set can divide without leaving a remainder. The LCM of 18, 24 and 30, for example, is 360.

Median  The median is a measure of central tendency. To find the median, we arrange the data from the smallest to largest value.

* If there is an odd number of data, the median is the middle value.
* If there is an even number of data, the median is the average of the two middle data points.

Mode  The value (data point) that occurs the most in a set of values (data) or is the data point with the largest frequency.

Multiple  The multiple of a certain number is that number multiplied by any other whole number. Multiples of 3, for example, are 6, 9, 12, 15, and so on.

Net  A plane shape that can be folded to make the solid.

Numerator  The part of the fraction that is written above the line. The 3 in \( \frac{3}{8} \), for example, is the numerator of the fraction. It also tells how many of that kind of fraction you have. In this case, you have 3 of them (eighths).

Orthogonal projection  A system of making engineering drawings showing several different views (for example, its plan and elevations) of an object at right angles to each other on a single drawing.

Parallel projection  Lines that are parallel in reality are also parallel on the drawing.

Pictogram (or pictograph)  Represents the frequency of data as pictures or symbols. Each picture or symbol may represent one or more units of the data.

Pie chart  A circular chart divided into sectors, where each sector shows the relative size of each value. In a pie chart, the angle of the each sector is in the same ratio as the quantity the sector represents.

Place value  Numbers are represented by an ordered sequence of digits where both the digit and its place value have to be known to determine its value. The 3 in 36, for example, indicates 3 tens and 6 is the number of units.

Satisfy  an equation, means that there is a certain value(s) that will make the equation true. In the equation \( 4x + 3 = -9 \), \( x = -3 \) satisfies the equation because \( 4(-3) + 3 = -9 \).

Simplify  means that you are writing an algebraic expression in a form that is easier to use if you want to do something else with the expression. If you want to add fractions, for example, you need to write all the fractions with the same denominator to be able to add them. Then the simplest form of \( \frac{3}{4} \) is \( \frac{9}{12} \), if 12 is the common denominator.
Simultaneous linear equations are equations that you solve by finding the solution that will make them simultaneously true. In $2x - 5y = 16$ and $x + 4y = -5$, $x = 3$ and $y = -2$ satisfy both equations simultaneously.

SI units The international system of units of expressing the magnitudes or quantities of important natural phenomena such as length in metres, mass in kilograms and so on.

Solve an equation means that we find the value of the unknown (variable) in the equation that will make the statement true. In the equation $3x - 4 = 11$, the value of the unknown (in this case, $x$) that will make the statement true, is 5, because $3(5) - 4 = 11$.

Terms in an algebraic expression are numbers and variables which are separated by + or − signs.

Variable. In algebra, variables are represented by letter symbols and are called variables because the values represented by the letter symbols may vary or change and therefore are not constant.

Vertex (plural vertices) A point where two or more edges of a solid meet.

$x$-axis The horizontal axis on a Cartesian plane.

$y$-axis The vertical axis on a Cartesian plane.

Teaching notes
You should be aware of what your class knows about the work from previous years. It would be good if you could analyse their answer papers from the previous end of year examination to determine where the class lacks the necessary knowledge and ability in previous work. You can then analyse the students’ answers to determine where they experience difficulties with the work, and then use this chapter to concentrate on those areas.

A good idea would be that you review previous work by means of the summary given in each section. Then you let the students do Review test 1 of that section and you discuss the answers when they finished it. You then let the students write Review test 2 as a test, and you let them mark it under your supervision.
Chapter 1: Numerical processes 1: Surds

Learning objectives
By the end of this chapter, the students should be able to:
1. Distinguish between rational numbers and non-rational numbers.
2. Identify numbers in surd form.
3. Simplify numerical surds.
4. Add, subtract, multiply and divide surds.
5. Rationalise the denominators of fractions involving surds.
7. Use conjugates to rationalise the denominator of surds with binomial fractions.
8. Express trigonometric ratios of 30°, 60°, 45° in terms of surds, and use them to calculate lengths in geometric figures and dimensions in real-life applications.
9. Apply surds to find lengths and distances in applications that involve Pythagoras’s theorem.

Teaching and learning materials
Teacher: Textbook and charts showing conjugates and the trigonometric ratios of 30°, 60° and 45° based on Figures 1.1 and 1.2.

Glossary of terms
Conjugate surds: the conjugate of \( \sqrt{x} + \sqrt{y} \) (where \( \sqrt{y} \) is irrational and \( x \) and \( y \) are rational) is \( x - \sqrt{y} \). Conjugate surds have the property that both their product \( (x^2 - y) \) and sum \( 2x \) are rational.

Teaching notes
• Explain the difference between rational and irrational numbers.
  ◾ You could stress the fact that there is a method to change an infinite recurring fraction to a common fraction. If, however, the digits after the decimal point of an infinite decimal fraction, shows no pattern of numbers that repeat, there is no method which can be used to write such a fraction as a common fraction. A fraction like this is called an irrational number.
  ◾ Examples of irrational numbers are any root that cannot be determined precisely (for example, \( \sqrt{5} \), \( \sqrt{7} \), \( \sqrt{8} \), and so on) and the most well-known irrational number, \( \pi \).

The simplest reason why pi is irrational: since \( \pi = \frac{C}{d} \) (where \( C \) = circumference of a circle and \( d \) = diameter of a circle), \( C \) can never be determined precisely.
So, \( C \) would contain an infinite non-recurring decimal fraction part and would, therefore, be irrational.

• Here are the surd laws:
  ◾ \( \sqrt{mn} = \sqrt{m} \times \sqrt{n} \) because from exponential laws \( (mn)^{\frac{1}{2}} = m^{\frac{1}{2}} \times n^{\frac{1}{2}} \)
  ◾ \( \sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}} \) because from exponential laws \( \left( \frac{m}{n} \right)^{\frac{1}{2}} = \frac{m^{\frac{1}{2}}}{n^{\frac{1}{2}}} \)
  ◾ \( a\sqrt{n} = \sqrt{a^2n} \)

• It is important that students remember that \( \sqrt{16} + 9 = 4 + 3 \) and \( \sqrt{25} - 9 = 5 - 3 \).

• Explain why, for example \( \sqrt{3} \times \sqrt{3} = 3 \) as follows:
  \( \sqrt{4} \times \sqrt{4} = 2 \times 2 = 4 \)
  \( \sqrt{9} \times \sqrt{9} = 3 \times 3 = 9 \) or \( \sqrt{3} \times \sqrt{3} = 3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3^{\frac{1}{2} + \frac{1}{2}} = 3 \)
  \( \therefore \sqrt{3} \times \sqrt{3} = 3 \)

• When we simplify surds by writing them with the smallest possible number under the root sign, we write them in this form because this form is the simplest if we want to add or subtract surds.
The reason is that we now create terms which are alike, for example, \( \sqrt{3} \)’s or \( \sqrt{5} \)’s and we can then add or subtract all the \( \sqrt{3} \)’s or \( \sqrt{5} \)’s.
• We rationalise the denominator (make it a rational number), because it then is easier to determine the value of a fraction (evaluate the fraction), because we do not have to divide by an irrational number.

• The principle we use to rationalise a fraction like \( \frac{\sqrt{5}}{\sqrt{2}} \) is that we multiply the numerator by \( \sqrt{2} \) and the denominator by \( \sqrt{2} \): \( \frac{\sqrt{5}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \). If we do this, we actually multiply by 1 and do not change the original fraction and this method works because \( \sqrt{2} \times \sqrt{2} = 2 \).

• When the denominator of a fraction contains two terms we use the difference of two squares to make the denominator a rational number.
  - So, if the denominator is \( 3\sqrt{2} - 4 \), for example, we multiply both the numerator and denominator by \( 3\sqrt{2} + 4 \).
  - If we do this, we actually multiply by 1 and do not change the original fraction and this method works because \( (3\sqrt{2} - 4)(3\sqrt{2} + 4) = (3\sqrt{2})^2 - (4)^2 \)
  \( = 9(2) - 16 = 2. \)

  The principle here is that \( (a - b)(a + b) = a^2 - b^2 \).

Areas of difficulty and common mistakes

• You can also tell the students that the 45° right-angled triangle has an interesting history. Pythagoras and his followers believed that they could describe anything in their world by numbers and that no calculation was impossible. When they discovered that they could not calculate the length of the hypotenuse of an isosceles right-angled triangle, they were shocked and tried to keep this a secret.

  One of Pythagoras’s followers, however, told this to other people. Legend has it that he was then drowned by some of Pythagoras’s followers.

  Students should remember the 45° and 30°/60° triangles by heart to be able to use these trigonometric ratios.

• Students tend to forget the 45° and 30°/60° triangles. Insist that they commit these triangles to their memories.

• Students could struggle to make the sketches of Exercise 1k and part of Exercise 1l. Teach them to read the questions very carefully and make the drawing sentence by sentence.
Learning objectives
By the end of this chapter, the students should be able to:
1. Express statements given in index form (such as \(81 = 3^4\)) as an equivalent logarithm statement (\(\log_3 81 = 4\)).
2. Evaluate expressions given in logarithmic form.
3. Note the equivalence between the laws of indices and the laws of logarithms.
4. Recall and use the laws of logarithms to simplify and/or evaluate given expressions without the use of logarithm tables.
5. Use logarithm tables for the purposes of calculation.

Teaching and learning materials
Teacher: Chart demonstrating laws of logarithms; logarithm tables (as used by Examinations Boards).

Glossary of terms
Logarithm of a number is the exponent to which another fixed value, the base, must be raised to produce that number. For example, the logarithm of 1000 to the base 10 is 3, because 10 to the power 3 is 1000: \(1000 = 10 \times 10 \times 10 = 10^3\).
Antilogarithm is the number of which a given number is the logarithm. For example, the antilogarithm of 3, if the base was 10, is 1000.
Evaluate means to determine the number value of something.

Teaching notes
* Note that, if \(a^x = b\) is the exponential form, then \(x = \log_a b\) is the logarithmic form. Also note that \(a > 0; a \neq 1; x > 0; x \text{ and } y \in \mathbb{R}\). This can be explained by doing these examples:
Determine:
a) \(\log_2 8\) b) \(\log_3 (-27)\)
c) \(\log_{-3} 27\) d) \(\log_3 0\)
Solution:
a) Let \(\log_2 8 = k\)
\(1^k = 8\), which is impossible, because 1 to any power always gives 1. So, that is why we say that \(a \neq 1\) in \(\log_a x\).
b) Let \(\log_3 (-27) = m\).
\(3^m = -27\), which is impossible, because a positive number to any power cannot give a negative answer. It can only give a positive answer. So, that is why we say that in \(\log_a x\), \(x > 0\).
c) Let \(\log_{-3} 27 = c\).
\((-3)^c = 27\), which is impossible. (In some cases, the base may be negative, but in Mathematics we want something to always be true: \(\log_{-3} 9 = c\), so \((-3)^c = 9\), then \(c = 2\).)
So, to have no doubts or exceptions, we say that in \(\log_a x\), \(a > 0\)
d) Let \(\log_3 0 = d\). Then \(3^d = 0\), which is impossible, because 3 to any power will never be equal to 0.
So that is why we say that in \(\log_a x\), \(x > 0\) and not \(\geq 0\).

* When students have to evaluate logs with different bases of easy examples (such as \(\log_5 4\)), teach them to ask themselves, “what exponent of 2 will give 4?”.
If the example is more difficult (such as \(\log_4 8\)), let them then make \(\log_4 8\) equal to \(x\) or any other variable.
Then, write the equation in exponential form and solve the exponent as shown in Example 1.
* Note that the purpose of Exercise 2b is to let the students practise using the logarithmic laws.
Areas of difficulty and common mistakes

• Students tend to become confused when they have to write the exponential form \( N = a^x \) (in the logarithmic form \( \log_a N = x \)), especially if the base is not 10 anymore. Give them enough examples to practise this.

• Students fail to notice the difference between, for example, \( \log 8 - \log 4 \) and \( \log 8 \div \log 4 \).
  - The first expression can be simplified, using the fact that \( \log M - \log N = \log \frac{M}{N} \).
    So, \( \log 8 - \log 4 = \log \frac{8}{4} = \log 2 \).
  - The second expression can also be simplified, using the fact that \( \frac{\log 8}{\log 4} = \frac{\log 2^3}{\log 2^2} = \frac{3\log 2}{2\log 2} = \frac{3}{2} = 1\frac{1}{2} \).
  - The same principle applies to \( \log 8 + \log 4 \) and \( \log 8 \times \log 4 \):
    \( \log 8 + \log 4 = \log (8 \times 4) = \log 32 \).
    The second expression cannot really be simplified without using log tables or a scientific calculator.
Learning objectives
By the end of this chapter, the students should be able to:
1. Solve quadratic equations using factorisation, completing the square, the quadratic formula and graphical methods.
2. Transform word problems into linear equations that may be solved algebraically.
3. Transform word problems into quadratic equations that may be solved algebraically.
4. Use algebraic and/or graphical methods to solve simultaneous linear and quadratic equations.

Teaching and learning materials
Students: Textbook, exercise book, writing materials, graph books or graph paper and aids for drawing curves (wire or twine).
Teacher: Graph boards or if an overhead projector is available, transparencies of graph paper, transparency pens and aids for drawing curves (wire or twine).

Teaching notes
• When quadratic equations are solved by means of factorisation, it is essential that students can actually factorise quadratic trinomials. If they still cannot do this, they will have to use the quadratic formula.
• When students solve quadratic equations using factorisation, teach them to always follow these steps:
  Step 1: Write the equation so that the right-hand side of the equation is equal to 0.
  Step 2: Simplify the left-hand side of the equation, so that there are only three terms: a term with the square of the variable, a term with just the variable and a constant.
  Step 3: Factorise the left-hand side.
  Step 4: Apply the principle “if $P \times Q = 0$, then $P = 0$ or $Q = 0$ or $P$ and $Q = 0$” and solve for the variable, for example, $x$.
• Teach students to only use the method of completing the square when specifically asked to use it.
• When using the quadratic formula, teach students to do the following:
  1. Rewrite the equation in the standard form of $ax^2 + bx + c = 0$.
  2. Write down the values of $a$, $b$ and $c$ as $a = \ldots$, $b = \ldots$, $c = \ldots$.
  3. Write down the quadratic formula:
     $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
  4. Then write brackets where the letters were:
     $$x = -(...) \pm \sqrt{(...)^2 - 4(...)(...) \over 2(...)$$
  5. Write the values of $a$, $b$ and $c$ into the brackets.
  6. Use a calculator to work out the answer.
• When students use the graphic method to solve a quadratic equation, teach them to draw the graph as accurate as possible.
  Also mention that the drawback of using a quadratic graph is that, in some cases, the roots of the quadratic equation can only be estimated.
  The graph, however, gives us a clear picture of why, in some cases, there are no roots, equal roots or two roots.
• When solving simultaneous linear and quadratic equations, work can be considerably more difficult if the linear equation $3x + y - 7 = 0$, is written as $x = \ldots$ instead of $y = 7 - 3x$.
  If we solve for $x$ in this case, we would work with a fraction ($x = {y - 7 \over 3}$).
  Teach the students to always, if possible, solve for the variable that will not result in working with a fraction.
• When solving quadratic equations where the variable is underneath the line, like \( \frac{r+1}{r+2} + \frac{r-3}{r-4} = 0 \), emphasise that \( r \neq -2 \) and \( r \neq 4 \), because division by 0 is undefined.

Areas of difficulty and common mistakes

• When solving quadratic equations such as \( x^2 + 4x = 21 \) or \( x(6x - 5) = 6 \) or \( x(x - 1) = 6 \) by factorisation, students do not realise that right-hand side of the equation must always be equal to 0. You must emphasise that we use the zero-product principle which states that if \( A \times B = 0 \), then \( A = 0 \) or \( B = 0 \) or both of them are equal to 0. If students solve the equation \( x(6x - 5) = 6 \) and write \( x = 6 \) or \( 6x - 5 = 6 \), it is completely incorrect, because then \( x(6x - 5) = 6 \times 6 = 36 \neq 6 \).

If students say that \( x = 1 \) and \( 6x - 5 = 6 \), that can also not be a correct method to solve the equation, because there are an infinite number of products which will give 6. For example, \( 3 \times 2, 12 \times \frac{1}{2}, 3 \times 4, \frac{1}{4} \times 24 \), and so on.

If the right-hand side is equal to zero, the possibilities are limited to 0 or 0.

• Students use the quadratic formula when they are asked specifically to complete the square to find the root(s) of a quadratic equation. Teach them to always do what is asked.

• When students solve quadratic equations and the method they must use is not specified, they tend to use completion of the square. By doing this, they make the work much more difficult for themselves. Teach them to do this:
  - If they cannot factorise the equation, then use the quadratic formula, even if the equation has factors.
  - They must only complete the square, if they are asked to do so.

• Students tend to make careless mistakes when solving simultaneous linear and quadratic equations.
  - Teach them to make ample use of brackets when substituting the value of one of the variables (obtained from the linear equation) into the quadratic equation.
  - Also teach them to check each step as they progress with their solution.

• Students find word problems that lead to linear or quadratic equations very difficult. You can do the following to help them:
  1. Use short sentences when you write a problem for them to solve.
  2. Ask them to write down their steps.
  3. If an area or perimeter is involved, tell them to make sketches of the problem.
  4. Sometimes the problem can also be made easier by representing it in table form and then completing the table with the information given in the problem. So, Example 7 can also be done like this:

**Lorry with …** | Distance | Speed | Time |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordinary speed</td>
<td>240</td>
<td>( x )</td>
<td>( \frac{240}{x} )</td>
</tr>
<tr>
<td>Slower speed</td>
<td>240</td>
<td>( x - 4 )</td>
<td>( \frac{240}{x - 4} )</td>
</tr>
</tbody>
</table>

\( \frac{240}{x - 4} - \frac{240}{x} = 2 \), and so on.

• Sometimes students find it difficult to work out the points that lie on the quadratic or linear graph. Teach them to use brackets when they substitute the values of \( x \) and to concentrate to work accurate.

• Sometimes students are not able to plot the points of the graphs accurately. Make sure that they do by helping those with problems individually.

Supplementary worked examples

1. Solve for \( x \):
   a) \( x - \frac{x + 3}{x} + \frac{3}{x} = 5 \)
   b) \( x + 5 + \frac{12}{x - 2} = 0 \)

2. a) Solve for \( p \), if \( p + \frac{40}{p} = 14 \).
   b) Now use your answer in (a) to solve for \( x \), if \( x^2 - 3x + \frac{40}{x(x - 3)} = 14 \).

Solution

1. a) \( (x)(x - (x)\frac{x + 3}{x} + (\frac{x}{x})^2 = 5(x) \)
   \( x^2 - (x + 3) + \frac{3}{x} = 5x \) (note the bracket)
   \( x^2 - x - 3 + 3 - 5x = 0 \) to avoid errors
   \( x^2 - 6x = 0 \)
   \( x(x - 6) = 0 \)
   \( x = 0 \) or \( x = 6 \), but \( x \neq 0 \) (we cannot divide \( \therefore x = 6 \) by 0)
   b) \( (x - 2)x + 5(x - 2) + (x - 2)\frac{12}{x - 2} = 0(x - 2) \)
   \( x^2 - 2x + 5x - 10 + 12 = 0 \)
   \( x^2 + 3x + 2 = 0 \)
   \( (x + 2)(x + 1) = 0 \)
   \( \therefore x = -2 \) or \( x = -1 \)
2. a) \( p^2 + 40 = 14p \)
\[ p^2 - 14p + 40 = 0 \]
\[ (p - 10)(p - 4) = 0 \]
\[ \therefore p = 10 \text{ or } p = 4 \]

b) \( x^2 - 3x = 10 \) or \( x^2 - 3x = 4 \)
\[ x^2 - 3x - 10 = 0 \]
\[ x^2 - 3x - 4 = 0 \]
\[ (x - 5)(x + 2) = 0 \]
\[ (x - 4)(x + 1) = 0 \]
\[ \therefore x = 5 \text{ or } x = -2 \]
\[ x = 4 \text{ or } x = -1 \]
Chapter 4: Mensuration 1: Length, area and volume

Learning objectives
By the end of this chapter, the students should be able to:
1. Calculate the length of a circular arc.
2. Calculate the area of sectors and segments of a circle.
3. Recall and use the formulae \( A = 4\pi r^2 \) and \( V = \frac{4}{3}\pi r^3 \) to calculate the area and volume of a sphere.
4. Calculate the outcomes when adding or subtracting compatible areas and volumes.
5. Solve problems involving change of shape of quantities such as liquids.
6. Use linear, area and volume factors \((d, d^2, d^3)\) when comparing the dimensions of shapes that are geometrically similar.

Teaching and learning materials

Students: Textbook, exercise book, writing materials, any flat shape or solid like tin cans, balls, boxes, and so on.
Teacher: Collection of plane and solid shapes, such as a ball, tins, boxes, and so on.

Teaching notes

• Remind the students of the area formula for triangles:
\[
\text{Area } \triangle ABC = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A
\]
This formula is used when you have two sides, and the angle between the two sides.

• In Example 1, the radius is equal to 3 m because:
\[
\text{In } \triangle OCA: \angle COA = 60^\circ \quad \text{(given)}
\]
But, \( OC = OA \) (both are radii of the circle)
So, \( \angle OCA = \angle OAC = \frac{180^\circ - 60^\circ}{2} = 60^\circ \).
This means that \( \triangle OCA \) is equilateral.
\[
\therefore \quad CA = OC = OA = 3 \text{ m}
\]

• In Example 2, \( \triangle OAM \cong \triangle OBM \) because:
- \( AM = MB \) (given)
- \( OM = OM \) (same side)
- \( OA = OB \) (radii of circle)
\[
\therefore \quad \triangle OAM \cong \triangle OBM \quad \text{(SSS = SSS)}
\]
\[
\therefore \quad \angle AOM = \angle BOM = \alpha
\]

• For change of shape you must assume that no metal or liquid is lost in the process. You can give students a summary of how the units of measurement are related. Below is an example of such a summary:

<table>
<thead>
<tr>
<th>Length</th>
<th>Area</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 cm = 10 mm</td>
<td>( 1 \text{ cm} \times 1 \text{ cm} = 10 \text{ mm} \times 10 \text{ mm} )</td>
<td>( 1 \text{ cm} \times 1 \text{ cm} \times 1 \text{ cm} = 10 \text{ mm} \times 10 \text{ mm} \times 10 \text{ mm} )</td>
</tr>
<tr>
<td>1 m = 100 cm</td>
<td>( 1 \text{ m} \times 1 \text{ m} = 100 \text{ cm} \times 100 \text{ cm} )</td>
<td>( 1 \text{ m} \times 1 \text{ m} \times 1 \text{ m} = 100 \text{ cm} \times 100 \text{ cm} \times 100 \text{ cm} )</td>
</tr>
<tr>
<td>1 m = 1 000 mm</td>
<td>( 1 \text{ m} \times 1 \text{ m} = 1 \text{ 000 mm} \times 1 \text{ 000 mm} )</td>
<td>( 1 \text{ m} \times 1 \text{ m} \times 1 \text{ m} = 1 \text{ 000 mm} \times 1 \text{ 000 mm} \times 1 \text{ 000 mm} )</td>
</tr>
<tr>
<td>1 km = 1 000 m</td>
<td>( 1 \text{ km} \times 1 \text{ km} = 1 \text{ 000 m} \times 1 \text{ 000 m} )</td>
<td>( 1 \text{ 000 m} \times 1 \text{ 000 m} \times 1 \text{ 000 m} )</td>
</tr>
<tr>
<td></td>
<td>( 1 \text{ km} \times 1 \text{ km} \times 1 \text{ km} = 1 \text{ 000 000 m} \times 1 \text{ 000 000 m} \times 1 \text{ 000 000 m} )</td>
<td>( 1 \text{ km} \times 1 \text{ km} \times 1 \text{ km} = 1 \text{ 000 000 000 m} \times 1 \text{ 000 000 000 m} \times 1 \text{ 000 000 000 m} )</td>
</tr>
</tbody>
</table>

Learning objectives
By the end of this chapter, the students should be able to:
1. Calculate the length of a circular arc.
2. Calculate the area of sectors and segments of a circle.
3. Recall and use the formulae \( A = 4\pi r^2 \) and \( V = \frac{4}{3}\pi r^3 \) to calculate the area and volume of a sphere.
4. Calculate the outcomes when adding or subtracting compatible areas and volumes.
5. Solve problems involving change of shape of quantities such as liquids.
6. Use linear, area and volume factors \((d, d^2, d^3)\) when comparing the dimensions of shapes that are geometrically similar.
• It is very important that the students know that, if the lengths of two similar shapes are in the ratio \(d:1\), the areas of corresponding faces are in the ratio \(d^2:1\) and their volumes are in the ratio \(d^3:1\). For example:

Length ratio of two similar shapes is in the ratio \(\frac{2}{3}:1\).

Areas of the faces of two similar shapes are in the ratio \(\left(\frac{2}{3}\right)^2 = \frac{4}{9}:1\).

Volumes of the two similar shapes are in the ratio \(\left(\frac{2}{3}\right)^3 = \frac{8}{27}:1\).

• It is useful to remember the following for prisms:

- Volume = Area of base \times height of prism
- Total surface area = 2 \times base area + perimeter of base \times height of prism.

You find the base of a prism if you imagine that you cut through it like you would cut bread and form congruent slices. Then the base is parallel to these cuts.

**Areas of difficulty and common mistakes**

• Students forget that they have to work with the same units. If, for example, the disk is 5 mm thick and has a diameter of 8 cm, either the 8 cm has to be converted to mm or the 5 mm has to be converted to cm.

• When students convert from the one unit to the other, they multiply or divide blindly without thinking what they are doing.
  - If they for example have to convert 256 m to km, they do not know whether they must divide by 1 000 or multiply by 1 000.
  - Teach them that if they convert from the smaller unit to the bigger unit, they have to divide.
  - If they have to convert from the bigger unit to the smaller one, they multiply because there are many of the smaller unit in the bigger one.

• When students have to convert the units of an area or the units of a volume to another unit, they usually struggle. This problem can be avoided if they use a table like the example above or if they convert the units before working out the area or volume.

• Students tend to want to only write down the answer when doing a problem.
  - Insist that they write all their work neatly down.
  - In this way, they can immediately see where an error is and correct it or you can see where they went wrong and correct a wrong thinking process.

**Supplementary worked examples**

1. The table on the next page gives information about enlargements of the parallelogram shown here.

   Copy the table in your book and complete it. (The area of a parallelogram is equal to base \((b)\) \times perpendicular height \((h)\)).

   ![Parallelogram diagram](image)

   - When students convert from the one unit to the other, they multiply or divide blindly without thinking what they are doing.
   - If they for example have to convert 256 m to km, they do not know whether they must divide by 1 000 or multiply by 1 000.
   - Teach them that if they convert from the smaller unit to the bigger unit, they have to divide.
   - If they have to convert from the bigger unit to the smaller one, they multiply because there are many of the smaller unit in the bigger one.
   - When students have to convert the units of an area or the units of a volume to another unit, they usually struggle. This problem can be avoided if they use a table like the example above or if they convert the units before working out the area or volume.
   - Students tend to want to only write down the answer when doing a problem.
     - Insist that they write all their work neatly down.
     - In this way, they can immediately see where an error is and correct it or you can see where they went wrong and correct a wrong thinking process.

   ![Table](image)

<table>
<thead>
<tr>
<th>Base length</th>
<th>Height of parallelogram</th>
<th>Length scale factor</th>
<th>Area scale factor</th>
<th>New area</th>
<th>Check your answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 cm</td>
<td>6 cm</td>
<td>1</td>
<td>1</td>
<td>48 cm²</td>
<td>(8 \times 6 = 48) cm²</td>
</tr>
<tr>
<td>18 cm</td>
<td>2</td>
<td></td>
<td></td>
<td>48 cm²</td>
<td></td>
</tr>
<tr>
<td>2.5 cm</td>
<td>100</td>
<td></td>
<td></td>
<td>(48 \times \ldots = )</td>
<td></td>
</tr>
<tr>
<td>320 cm</td>
<td>240 cm</td>
<td></td>
<td></td>
<td>(48 \times \ldots = )</td>
<td>(48 \times \ldots = 1 200) cm²</td>
</tr>
</tbody>
</table>
2. This table shows the volume of the cuboid shown here. Copy and complete the table.

![Cuboid diagram]

<table>
<thead>
<tr>
<th>Breadth</th>
<th>Length</th>
<th>Height</th>
<th>Length scale factor</th>
<th>Volume scale factor</th>
<th>New volume</th>
<th>Check your answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 cm</td>
<td>6 cm</td>
<td>4 cm</td>
<td>1</td>
<td>1</td>
<td>72 cm³</td>
<td>3 × 6 × 4 = 72 cm³</td>
</tr>
<tr>
<td>6 cm</td>
<td>12 cm</td>
<td>8 cm</td>
<td>2</td>
<td>8</td>
<td>72 cm³</td>
<td>6 × 12 × 8 = 576 cm³</td>
</tr>
<tr>
<td>9 cm</td>
<td>18 cm</td>
<td>12 cm</td>
<td>3</td>
<td>27</td>
<td>72 cm³</td>
<td>9 × 18 × 12 = 1 944 cm³</td>
</tr>
<tr>
<td>15 cm</td>
<td>30 cm</td>
<td>20 cm</td>
<td>5</td>
<td>125</td>
<td>72 cm³</td>
<td>15 × 30 × 20 = 9 000 cm³</td>
</tr>
<tr>
<td>18 cm</td>
<td>36 cm</td>
<td>24 cm</td>
<td>6</td>
<td>216</td>
<td>72 cm³</td>
<td>18 × 36 × 24 = 15 552 cm³</td>
</tr>
</tbody>
</table>

### Solution

1.

<table>
<thead>
<tr>
<th>Base length</th>
<th>Height of parallelogram</th>
<th>Length scale factor</th>
<th>Area scale factor</th>
<th>New area</th>
<th>Check your answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 cm</td>
<td>6 cm</td>
<td>1</td>
<td>1</td>
<td>48 cm²</td>
<td>8 × 6 = 42 cm²</td>
</tr>
<tr>
<td>16 cm</td>
<td>12 cm</td>
<td>2</td>
<td>4</td>
<td>48 cm²</td>
<td>16 × 12 = 192 cm²</td>
</tr>
<tr>
<td>24 cm</td>
<td>18 cm</td>
<td>3</td>
<td>9</td>
<td>48 cm²</td>
<td>24 × 18 = 432 cm²</td>
</tr>
<tr>
<td>2 cm</td>
<td>1½ cm</td>
<td>1/4</td>
<td>1/16</td>
<td>48 cm²</td>
<td>2 × 1½ = 3 cm²</td>
</tr>
<tr>
<td>80 cm</td>
<td>60 cm</td>
<td>10</td>
<td>100</td>
<td>48 cm²</td>
<td>80 × 60 = 4 800 cm²</td>
</tr>
<tr>
<td>40 cm</td>
<td>30 cm</td>
<td>5</td>
<td>25</td>
<td>48 cm²</td>
<td>40 × 30 = 1 200 cm³</td>
</tr>
<tr>
<td>320 cm</td>
<td>240 cm</td>
<td>40</td>
<td>1 600</td>
<td>48 cm²</td>
<td>320 × 240 = 76 800 cm³</td>
</tr>
</tbody>
</table>

2.

<table>
<thead>
<tr>
<th>Breadth</th>
<th>Length</th>
<th>Height</th>
<th>Length scale factor</th>
<th>Volume scale factor</th>
<th>New volume</th>
<th>Check your answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 cm</td>
<td>6 cm</td>
<td>4 cm</td>
<td>1</td>
<td>1</td>
<td>72 cm³</td>
<td>3 × 6 × 4 = 72 cm³</td>
</tr>
<tr>
<td>6 cm</td>
<td>12 cm</td>
<td>8 cm</td>
<td>2</td>
<td>8</td>
<td>72 cm³</td>
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</tr>
<tr>
<td>9 cm</td>
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<td>125</td>
<td>72 cm³</td>
<td>15 × 30 × 20 = 9 000 cm³</td>
</tr>
<tr>
<td>18 cm</td>
<td>36 cm</td>
<td>24 cm</td>
<td>6</td>
<td>216</td>
<td>72 cm³</td>
<td>18 × 36 × 24 = 15 552 cm³</td>
</tr>
</tbody>
</table>
Learning objectives
By the end of this chapter, the students should be able to:
1. Calculate the simple and compound interest on a given principal at a given rate over a given period of time.
2. Use the compound interest formula, \( A = P\left(1 + \frac{r}{100}\right)^n \) to calculate the amount A, if a principal P is invested for \( n \) years at \( r \)% per annum.
3. Calculate the depreciation as an item loses value over a given period of time.
4. Calculate how prices are affected by inflation over time.
5. Calculate the dividend arising from investing in stocks and shares.
6. Calculate the financial benefits in purchasing an annuity.
7. Calculate the income tax levied on given incomes.
8. Determine the value-added tax (VAT) paid on certain goods and services.

Teaching and learning materials
Students: Textbook, exercise book, writing materials, logarithm tables (see pages 272–273 of textbook) and a scientific calculator.
Teacher: Financial pages of newspapers and magazines that refer to interest, savings, stocks, shares, annuities, current income tax and VAT rates, examples of receipts that show VAT.

Teaching notes
• Note that the total amount, after simple interest can be calculated by using the formula
\[ A = P + \frac{PRT}{100} = P(1 + \frac{RT}{100}) \]
where \( A \) represents the amount after interest is added, \( P \) represents the amount invested (principal), \( R \) represents the interest rate per/\( \text{uni} \)year and \( T \) the time in/\( \text{uni} \)years. This formula could also be written as
\[ A = P(1 + in) \]
where \( i = \frac{r}{100} \) and \( r \) is the interest rate per year and \( n \) represents the number of years.

• You could explain to the students that compound interest means that one earns interest on the increased amount of money as well. You can then explain as follows how this compound increase works:
Let’s say that \( \text{R}1\,000 \) is invested at 12% interest per year.

After 1 year, the amount is increased by 12%:
\[ A_1 = 1\,000\left(1 + \frac{12}{100}\right) \]
\[ = 1\,000(1.12) \]
After 2 years, the amount at the end of the previous year is increased by 12%:
\[ A_2 = 1\,000(1.12)\left(1 + \frac{12}{100}\right) \]
\[ = 1\,000(1.12)(1.12) \]
\[ = 1\,000(1.12)^2 \]
After 3 years, the amount at the end of the previous year is increased by 12%:
\[ A_3 = 1\,000(1.12)^2\left(1 + \frac{12}{100}\right) \]
\[ = 1\,000(1.12)^2(1.12) \]
\[ = 1\,000(1.12)^3 \]
So, after \( n \) years, \( A_n = 1\,000(1.12)^n \) (where \( n \) = the number of years the money was invested at an interest rate of 12% per year).

• If the future or end amount = \( A \), the amount invested = \( P \) and \( i = \frac{r}{100} \) where \( r \) = interest rate per year, then the formula for compound interest could also be: \( A = P(1 + i)^n \), where \( n \) is the number of years the money was invested.

• It is good to be aware of the fact that there are two kinds of depreciation:
1. Linear depreciation (more or less like simple interest). The formula is:
\[ A = P(1 - in) \]
where \( A \) is the depreciated value or book value, \( P \) is the present value, \( i \) is the rate of depreciation per year or \( \frac{r}{100} \), \( n \) = time in years.
2. Reducing-balance depreciation where the depreciation is also a depreciation on the previous depreciated value. It can be compared to compound interest. The formula is:

\[ A = P(1 - \Delta)^n, \]

where \( A \) is the depreciated value or book value, \( P \) = present value, \( \Delta \) is the rate of depreciation per year or \( \frac{\Delta}{100} \) and \( n = \text{time in years} \).

- Inflation is always the increase on already increased prices or goods. So, inflation is always calculated like compound interest.
- Example 6 can also be done like this:

\[ A = P(1 + \Delta)^n \]

\[ = 800 000 (1 + 0.06)^3 \]

\[ = 800 000 (1.06)^3 \]

\[ = N\text{952 813} \]

The exponent is 3 because there are three periods where the interest rate is 6% per period.
- It could also be useful to know the following, especially if your more talented students want to know how banks usually calculate the value of somebody’s investment:

Let us say that N800 000 is invested at 12% per year for 1.5 years and the interest is compounded daily.

Now the interest rate per day = \( \frac{12\%}{365} \) and in 1.5 years there are 545 periods of time, if we assume that the money was invested on 1 January and that the years are not leap years.

So, \( A = 800 000 (1 + \frac{0.12}{365})^{545} \)

\[ = N\text{956 950} \] to the nearest N.

Note that we assume that the interest rate stays the same.
- If the interest rate is given per year, then the half yearly interest rate is half of the interest rate per year. So, if the interest rate per annum (year) is given as 10%, then the interest rate per half year would be 5%. (See number 18 or Exercise 5a.)
- Example 11 can be explained more fully like this:

The first amount paid at the end of the 1st year would yield interest for 2 years: N\text{10 000(1.08)}^2.

The second amount paid at the end of the 2nd year would yield interest for 1 year: N\text{10 000(1.08)}^1.

The last amount paid at the end of the 3rd year would yield no interest: N\text{10 000}.

:\: Total amount

\[ = N\text{10 000(1.08)^2 + N\text{10 000(1.08)^1 + N\text{10 000.}} \]

- When money is invested as a future value annuity, it means that it is money that will be received in the future. It usually is a way to save for retirement.

Let us say that Mr Okoro pays N24 000 per year to save for his retirement.

He pays this amount at the end of each year for 30 years.

Let us assume that the interest rate during that time stays 15% and is compounded annually.

The amount paid at the end of the first year will earn interest for 29 years: \( A_1 = 24 000 (1.15)^{29} \)

The amount paid at the end of the 2nd year will earn interest for 28 years: \( A_2 = 24 000 (1.15)^{28} \)

The amount paid at the end of the 3rd year will earn interest for 27 years: \( A_3 = 24 000 (1.15)^{27} \)

Expanding, we get:

\[ A_3 = 24 000 (1.15)^{27} + 24 000 (1.15)^{26} + \ldots + 24 000 (1.15)^1 \]

This is a geometric series with first term \( a = 24 000 \), the common ratio \( r = 1.15 \) and the number of terms \( n = 30 \).

\[ S_n = \frac{a(r^n - 1)}{r - 1} = \frac{24 000 (1.15^{30} - 1)}{1.15 - 1} = \frac{24 000 (1.15^{30} - 1)}{0.15} = N\text{10 433 883.51} \]

- When somebody buys a house, for example, the money borrowed, is paid back in instalments. This is called a present value annuity, because the amount of money to buy the house is immediately available and has to be paid back over time with interest.

Let us say that somebody pays instalments of N20 000 per month and we want to know how much he paid for the house over 20 years, if the interest rate stays a constant 12% per year.

1. The interest rate per month = 1%.
2. The number of periods = 20 × 12 = 240.
3. The N20 000 includes the interest and we want to know what the actual amount per month without interest is, because we want to know what was originally borrowed.
So, we have to work out \( P \) in \( A = P(1 + \dot{i})^n \):
\[
P = \frac{A}{(1 + \dot{i})^n}
\]
\[
P = 20 000(1.01)^{-1} + 20 000(1.01)^{-2} + 20 000(1.01)^{-3} + \ldots + 20 000(1.01)^{-20},
\]
where \( P \) is the present value or the total amount of money borrowed.

Now this also is a geometric series with first term \( a = 20 000(1.01)^{-1} \), the common ratio \( r = (1.01)^{-1} \) and the number of terms \( n = 240 \).

Answer: \( \text{₦}1,816,388.33 \) or this is the amount borrowed. If the person, however, paid \( \text{₦}20,000 \) per month, the total amount paid for the house = \( \text{₦}20,000 \times 240 = \text{₦}4,800,000! \)

So, the total interest paid over the 20 years = \( \text{₦}1,983,612. \)

Areas of difficulty and common mistakes

- Working out the percentage yield per share.
  Here the student must remember that each share was bought for a certain amount and that the dividend is an amount per share.
  So, if the dividend is 10k per share, it means that for each share that the investor owns of that specific company, he or she gets 10 kobo.
  So, if the investor wants to work out the percentage yield per share, if he originally paid \( \text{₦}1.08 \) per share, for example, he must do the following:

  Percentage earned on each share
  \[
  = \frac{10k}{108k} \times 100 = 9.3\%.
  \]

  A dividend is always an amount per share. So to work what % the investment yielded, you take the amount earned per share, divide it by the amount you originally paid per share and multiply this fraction by 100.

- Finding the amount paid for VAT if the final amount, which includes VAT is given. Now this amount is the already increased amount. So you must tell the students to remember that this final amount is 105% (if VAT is 5%) of the original amount without VAT. So this final amount must be reduced to 100%.

- Working out income tax. The taxable amount must always be broken down into the tax bands and the tax for each band worked out and added to give the total income tax.

### Supplementary worked examples

Steven wants to save \( \text{₦}20,000 \) over a period of 2 years.

If the interest rate is 15% per year and will stay the same, what amount of money must he save each month for the next 2 years?

Steven starts saving at the end of the first month.

**Solution:**

1. The interest rate per month = 1.25% = 0.0125
2. The number of periods = \( 12 \times 2 = 24 \).
   
   The 1st amount, \( A_1 = x(1 + 0.0125)^{24} = x(1.0125)^{24} \)
   
   The 2nd amount, \( A_2 = x(1,0125)^{23} \)
   
   The 3rd amount, \( A_3 = x(1,0125)^{22} \)
   
   The last amount would not earn any interest.
   
   \[ \therefore \text{₦}20,000 = x(1.0125)^{23} + x(1,0125)^{22} + \ldots + x(1,0125)^{24} \]

   This is a geometric series with first term \( a = x \), the common ratio \( r = 1.0125 \) and the number of terms \( n = 24 \).

   \[
   \frac{a(r^n - 1)}{r - 1} = 20,000
   \]

   \[
   x = \frac{20,000 	imes 0.0125}{(1.0125^{24} - 1)} = 719.73
   \]

   Answer = \( \text{₦}719.73 \) per month, or \( \text{₦}720 \) to the nearest \( \text{₦} \).
Learning objectives
By the end of this chapter, the students should be able to:
1. Determine the sine, cosine and tangent of any angle between 0° and 360°.
2. Given sin θ, cos θ and tan θ, determine θ, where 0° ≤ θ ≤ 360°.
3. Use the unit circle to develop graphs for sin θ, cos θ and tan θ, for 0° ≤ θ ≤ 360°.
4. Sketch the graphs of the form A ± B sin nθ for simple numerical values of A, B and n.

Teaching and learning materials
Students: Textbook, exercise book, writing materials, graph books or graph paper and aids for drawing curves (wire or twine).
Teacher: Graph boards, aids for drawing curves (wire or twine). If an overhead projector is available: transparencies of graph paper and transparency pens.

Teaching notes
* Make sure that the students understand the fact that \( \sin \theta = \frac{y}{r} \), \( \cos \theta = \frac{x}{r} \) and \( \tan \theta = \frac{y}{x} \) and that the size of θ determines the values of the x- and y-coordinates and that the value of r is always positive.

Then explain the following:
* In the second quadrant, the x-coordinate is negative; so all the trigonometric ratios that contain an x-coordinate will be negative. They are the cos and tan ratios.

The sin-ratio contains a y-coordinate and is, therefore, positive in the second quadrant. In the sketch, A and A’ are symmetrical around the y-axis. So, all the angles in the second quadrant can be written as 180° − θ, where θ is an acute angle. For example:

\[
\sin 120° = \sin (180° - 60°) = \sin 60°
\]
\[
\cos 120° = \cos (180° - 60°) = -\cos 60°
\]
\[
\tan 120° = \tan (180° - 60°) = -\tan 60°
\]

* In the third quadrant, both the x- and y-coordinates are negative; so all the trigonometric ratios that contain either an x-coordinate or a y-coordinate, will be negative. They are the sin and cos ratios.

The tangent-ratio contains both an x- and a y-coordinate; and since a negative number divided by a negative number gives a positive answer, the tangent-ratio is positive in the third quadrant. In the sketch, A and A’, are symmetrical around the origin. So, the angles in the third quadrant can all be written as 180° + θ, where θ is an acute angle. For example:

\[
\sin 240° = \sin (180° + 60°) = -\sin 60°
\]
\[
\cos 240° = \cos (180° + 60°) = -\cos 60°
\]
\[
\tan 240° = \tan (180° + 60°) = \tan 60°
\]
• In the fourth quadrant, the y-coordinates are negative; so all the trigonometric ratios that contain a y-coordinate will be negative. They are the sin and tan ratios.

![Diagram of trigonometric ratios in the fourth quadrant](image)

The cos-ratio contains an x-coordinate and will be positive in the fourth quadrant.

In the sketch, A and A’ are symmetrical around the x-axis.

So, the angles in the fourth quadrant can all be written as 360° − θ where θ is an acute angle.

For example:
- \( \sin 300° = \sin (360° - 60°) = -\sin 60° \)
- \( \cos 300° = \cos (360° - 60°) = \cos 60° \)
- \( \tan 300° = \tan (360° - 60°) = -\tan 60° \)

• When you explain to the students how to draw sketch graphs of \( y = \sin \theta \) and \( y = \cos \theta \), explain that they only have to show where these graphs intersect the x- and y-axes, and the coordinates of their turning points.

So, to draw these graphs students can use the unit circle, and the points where it intersects the x- and y-axes:

<table>
<thead>
<tr>
<th>Angle</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>90°</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>180°</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>270°</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>360°</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

So, explain to the students that the sign of the trigonometric ratio depends entirely on the quadrant in which the radius of the circle falls.

• When the sine and cosine graphs are seen as waves, students can also regard:
  - Their maximum deviation from the x-axis as the amplitude of the wave, which, in these cases, is 1.
  - One repetition of the graph as its period and for \( y = \sin \theta \) and \( y = \cos \theta \) the period is 360°.

• Tell the students that in \( y = m \cdot \cos \theta \) or \( y = m \cdot \sin \theta \), the “\( m \)” represents the amplitude of the graph. So, if they have to draw a graph like, for example, \( y = 2 \cdot \sin \theta \) or \( y = \frac{1}{2} \cdot \cos \theta \), the maximum deviation (amplitude) from the x-axis is 2 and \( \frac{1}{2} \) respectively: \( y = \cos \theta \) and \( y = \frac{1}{2} \cdot \cos \theta \)
When students have to draw a graph of \( y = \sin 3\theta \), for example, tell them that they can again use the unit circle, but that they now have to find the values of \( \theta \), if the values of \( 3\theta \) are 0°, 90°, 180°, 270° and 360° to find one period (or repetition) of the graph as shown in this table:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0°</th>
<th>30°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3\theta )</td>
<td>0°</td>
<td>90°</td>
<td>180°</td>
<td>270°</td>
<td>360°</td>
</tr>
<tr>
<td>( y = \sin 3\theta )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>−1</td>
<td>0</td>
</tr>
</tbody>
</table>

The remainder of the graph then is just a repetition of the pattern of one period, and can easily be drawn as shown below:

When students have to draw a graph of \( y = \cos \frac{1}{2}\theta \), for example, tell them that they can again use the unit circle, but that they now have to find the values of \( \theta \), if the values of \( \frac{1}{2}\theta \) are 0°, 90°, 180°, 270° and 360° to find one period (or repetition) of the graph as shown in this table:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0°</th>
<th>180°</th>
<th>360°</th>
<th>540°</th>
<th>720°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2}\theta )</td>
<td>0°</td>
<td>90°</td>
<td>180°</td>
<td>270°</td>
<td>360°</td>
</tr>
<tr>
<td>( y = \cos \frac{1}{2}\theta )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>−1</td>
<td>0</td>
</tr>
</tbody>
</table>

As before, students will find it easy to complete the remainder of the pattern of the graph.

Explain to the students that they can find the period of:
- \( y = \sin ax \) and \( y = \cos ax \) (the number of degrees for one repetition of the graph) by dividing 360° by \( a \).
- They can find the new period of \( y = \tan ax \) by dividing 180° by \( a \).
  If, for example, \( y = \tan \frac{1}{2}x \), then the new period is \( 180° \times \frac{1}{2} = 180° \times 2 = 360° \).

When students have to draw graphs of \( y = a + \sin \theta \) or \( y = a + \cos \theta \), explain to them that it simply means that these graphs are translated \( a \) units upwards (if \( a > 0 \)) or \( a \) units downwards (if \( a < 0 \)).

In trigonometry, the Greek letter \( \theta \) (theta) is often used to indicate the size of an unknown angle. Other Greek letters often used to indicate sizes of angles in trigonometry are \( \alpha \) (alpha), \( \beta \) (beta) and \( \varphi \) (phi).

You can explain to the students that from the unit circle they should be able to see the following:
When \( \theta = 0° \), \( \tan 0° = \frac{y}{x} = 0 = 0 \)
When \( \theta = 90° \), \( \tan 90° = \frac{y}{x} = \frac{1}{0} \), which is not defined.
When \( \theta = 180° \), \( \tan 180° = \frac{y}{x} = \frac{0}{1} = 0 \)
When \( \theta = 270° \), \( \tan 270° = \frac{y}{x} = \frac{1}{0} \), which is not defined.
When \( \theta = 360° \), \( \tan 360° = \frac{y}{x} = \frac{0}{1} = 0 \).

You could also explain to them that, if they try to determine \( \tan 90° \) or \( \tan 270° \), by using scientific calculators, their calculators would give something like “Error 2”.

Chapter 6: Trigonometry: Angles between 0° and 360°
• If scientific calculators are available, you could let your students investigate what happens to the tangent ratio around 90° and 270° if they answer these questions:

1. Determine:
   a) \( \tan 89.9° \)
   b) \( \tan 89.99° \)
   c) \( \tan 89.999° \)
   d) \( \tan 89.9999° \)
   e) \( \tan 89.99999° \)
   f) What do you notice?

2. Determine:
   a) \( \tan 90.1° \)
   b) \( \tan 90.01° \)
   c) \( \tan 90.001° \)
   d) \( \tan 90.0001° \)
   e) \( \tan 90.00001° \)
   f) What do you notice?

3. Determine:
   a) \( \tan 269.9° \)
   b) \( \tan 269.99° \)
   c) \( \tan 269.999° \)
   d) \( \tan 269.9999° \)
   e) \( \tan 269.99999° \)
   f) What do you notice?

4. Determine:
   a) \( \tan 270.1° \)
   b) \( \tan 270.01° \)
   c) \( \tan 270.001° \)
   d) \( \tan 270.0001° \)
   e) \( \tan 270.00001° \)
   f) What do you notice?

5. Why can you not find \( \tan 90° \) or \( \tan 270° \)?
   • You can then explain the following to the class:
     ▪ If \( \theta \) increases from 0° to 90°, \( \tan \theta \) increases from 0 to an infinitely large positive number.
     ▪ If \( \theta \) increases from 90° to 180°, \( \tan \theta \) increases from an infinitely big negative number to 0.
     ▪ If \( \theta \) increases from 180° to 270°, \( \tan \theta \) increases from 0 to an infinitely big number.
     ▪ If \( \theta \) increases from 270° to 360°, \( \tan \theta \) increases from an infinitely big negative number to 0.
     ▪ \( \tan \theta \) is undefined at 90° and 270°, because at 90° or at 270° the \( x \)-coordinate is equal to 0 and division by 0 is undefined.
     ▪ Since \( \tan 90° \) and \( \tan 270° \) are not defined, we say that the graph has **asymptotes** there. These asymptotes are shown by a vertical broken line through 90° and 270°.

   ▪ An **asymptote** is a line towards which the graph comes nearer and nearer but can never intersect.

   ▪ The graph of the **tan function**, therefore, consists of separate parts, and we say that this function is **discontinuous** because it is **not defined for all values of the angle** \( \theta \).

   ▪ If you look at the graph you drew, you will see that the graph repeats itself every 180°. We, therefore, say that the **tan function** has a **period of 180°**.

   ▪ Lastly you can explain to your students that for a graph of the **tan-function** they need to show the following:
     - The asymptotes with a vertical broken line at 90°, 270°, …
     - Where the graph intersects the \( x \)-axis (the horizontal axis).
     - Where the graph intersects the \( y \)-axis (the vertical axis).
     - The coordinates of the points where \( \tan \theta = 1 \), for example, (45°, 1).
     - The coordinates of the points where \( \tan \theta = -1 \), for example, (135°, −1).

   ▪ When students draw the graph of \( y = \tan 2\theta \), they have to find the values of \( \theta \), if the values of \( 2\theta \) are 0°, 45°, 90°, 135°, 180° to find one period (or repetition) of the chart as shown in this table:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0°</th>
<th>45°</th>
<th>90°</th>
<th>135°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2\theta )</td>
<td>0°</td>
<td>22.5</td>
<td>45°</td>
<td>67.5°</td>
<td>90°</td>
</tr>
<tr>
<td>( y = \tan 2\theta )</td>
<td>0</td>
<td>1</td>
<td>Not defined</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

![Graph of tan function](image-url)
• Explain to the students that these graphs are examples of graphs of periodic functions. A periodic function is when its graph repeats itself after a certain period.

• Explain to the students that, when we solve an algebraic equation, we are looking for the values(s) of the variable that will satisfy the equation. When we solve a trigonometric equation, we must find the angle(s) that will satisfy the equation. If for example, the equation \( \cos x = -0.5 \) has to be solved, we can see from the graph that \( x = 120^\circ \), if \( \cos x = -0.5 \).

But if we look at the graph of \( y = \cos x \), we see that \( 120^\circ \) is only one of the solutions of the equation.

The other solution of \( 0^\circ \leq x \leq 360^\circ \) is \( 240^\circ \), because \( y = -0.5 \) intersects the graph of \( y = \cos x \) twice.

You can explain the other trigonometric equations also with the help of graphs.

**Areas of difficulty and common mistakes**

• If students use scientific calculators to solve a trigonometric equation such as \( \cos \theta = -0.2834 \), they tend to not first take the positive value of the cos ratio and first work out the angle in the first quadrant and then work out the angles in the 2\(^{nd}\) and 3\(^{rd}\) quadrants. Emphasise that they immediately write the following if, for example the cos-ratio is negative:

  **Step 1:** \( \theta \) is in the second quadrant and the third quadrant

  **Step 2:** Write down \( \theta = 180^\circ - \ldots \) or \( \theta = 180^\circ + \ldots \).

  **Step 3:** Now find the acute angle, if \( \cos \theta = 0.2834 \).

  **Step 4:** Now complete Step 2 by writing:

  \( \theta = 180^\circ - 73.5^\circ \) or \( \theta = 180^\circ + 73.5^\circ \) and complete your solution by writing down the values of \( \theta \).

• Students may find it difficult to draw sketch graphs of combinations, for example, not only \( y = \cos 2x \), but \( y = 1 + \cos 2x \) or not only \( y = 2\sin x \), but \( y = 2\sin 3x \).

Then teach them to first draw, for example \( y = \cos 2x \).

Then translate that graph 1 unit upwards, or first draw the graph of \( y = \sin 3x \) and then make the amplitude of the graph double its previous value. So, let them draw the graphs in steps.
Learning objectives
By the end of this chapter, the students should be able to:
1. Distinguish between great circles and small circles on a spherical surface.
2. Recall and use the definitions of lines of longitude (including the Greenwich Meridian) and latitude (including the equator) on the surface of the Earth.
3. Determine and sketch the position of a point on the surface of the Earth in terms of its latitude and longitude, for example, (14°N, 26°E) or (37°S, 105°W).
4. Calculate the distance between two points on a great circle (meridian and equator).
5. Calculate the distance between two points on a parallel latitude.
6. Calculate the speed of a point on the surface of the Earth due to the Earth's rotation.
7. Compare great-circle and small-circle routes on the surface of the Earth.

Teaching and learning materials
Teacher: Globe of the Earth, a skeletal model of a sphere with at least three great circles (including the equator) and at least one circle of latitude, potter’s clay to make a globe and fishing line to cut through the clay, a 360° protractor and a transparency pen to draw on the clay.

Teaching notes
• If you want the students to understand how latitude lines work, you must show them by making a model.
  ▪ Make a sphere of potter’s clay.
  ▪ Then, cut it into two hemispheres using a piece of fishing line or a thin wire.
  ▪ Now, find the approximate centre of the circles of the flat sides of the hemispheres.
  ▪ Then, use a protractor (preferably a 360° protractor), mark lines as shown in the sketch above and lengthen and draw the lines along the curved sides of the hemispheres.
  ▪ Now, explain to the students that these lines are the latitude lines and that the line along which you cut with the fishing line, represents the Prime Meridian or Greenwich Meridian.
  ▪ You could also explain to the students that these latitude lines are like the rungs of a ladder and run from east to west and tell how far up (North) one can go and how far down (South) one can go.
    ▪ These lines form circles around the earth.
    ▪ These circles become smaller the nearer they are to the poles of the Earth.
• To explain how longitude lines work, you can make another sphere out of potter’s clay, cut it into two hemispheres and use a protractor (preferably a 360° protractor) to mark lines as shown in the sketch below. You can then explain to the students that all these lines come together at the poles, and give the distance East or West of the Greenwich Meridian.
Areas of difficulty and common mistakes

- Many students find it extremely difficult to see 3-dimensionally from a sketch.
  - Use models of wire or the clay model above to explain better.
  - Then, make the sketch which represents the 3-dimensional situation and from that the plane view.
  - Always go back to the real model, if students still find it difficult to picture the situation.
- Students may find it difficult to understand how to find the radius of a parallel latitude.
  - To explain this better, a wire model of a ball is ideal.
  - You can use Figure 7.21a to give you an idea of how to make such a model that you can use to explain how to find the radius of the smaller circle.

- Explain to the students that longitude lines form great circles, but latitude lines form circles that become smaller as one goes up (North) from the equator or down (South) of the equator.
**Learning objectives**

By the end of this chapter, the students should be able to:

1. Organise data and information in a rectangular array, or matrix of rows and columns.
2. Add and subtract matrices of order up to $2 \times 2$.
3. Multiply matrices by a scalar.
4. Multiply two matrices of order up to $2 \times 2$.
5. Transpose a matrix.
6. Find the determinant of a $2 \times 2$ matrix.
7. Find the inverse of a $2 \times 2$ matrix.
8. Apply $2 \times 2$ matrix algebra to solve simultaneous equations.

**Teaching and learning materials**

**Students:** Textbook, exercise book and writing materials.

**Teacher:** Charts showing matrix addition and multiplication; examples of matrices from newspaper and magazine articles, matrix related computer instructional materials where available.

**Teaching notes**

- Explain to the students that, when we write down the order of a matrix, we first write the number of rows (the horizontal) and then the number of columns (vertical). This is similar to writing down the coordinates of a point.
  - We first write down the horizontal component ($x$-coordinate).
  - We then write down the vertical component ($y$-coordinate).

- Emphasise that matrices can only be added or subtracted if they are of the same order, which means that they must have exactly the same number of rows and columns.

- Explain to the students that the transpose of a **square matrix** can also be found by reflecting the entries of the matrix over the main diagonal. This will only work for square matrices, however.

- If $A = \begin{bmatrix} 1 & 4 \\ 2 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, then $A \times B = \begin{bmatrix} 1 \times (-2) + 4 \times (3) \\ 2 \times (-2) + (-7) \times (3) \end{bmatrix} = \begin{bmatrix} 10 \\ -25 \end{bmatrix}$.

- Cramer’s rule uses determinants to solve systems of linear equations. This rule is especially useful if the answers contain complicated fractions.

If we multiply two matrices $A$ and $B$ by each other, we multiply each entry of the first row of $A$ with each entry of the first column of $B$ and add the products. That gives us the first entry of the product of the two matrices.

Then, we multiply each entry of the second row of $A$ with each entry of the first column of $B$ and add the products. That again gives us the second entry of the product of the two matrices.

$A$ is a $2 \times 2$ matrix and $B$ is a $2 \times 1$ matrix.

For multiplication of these two matrices to be possible, the number of rows of $A$ must be equal to the number of columns of $B$. The product is a $2 \times 1$ matrix. This means that:

- The dimensions of $A \times B = 2 \times 1$.

- Cramer’s Rule states that:

  $x = \frac{rd - bs}{ad - bc}$

  and $y = \frac{rc - as}{ad - bc}$. 

  If we want to solve for $x$ and $y$ in $ax + by = r$ and $cx + dy = s$, Cramer’s Rule states that $x = \frac{rd - bs}{ad - bc}$.
From this, we can see that the numerator of the \( x \)-value is the determinant of the matrix \( \begin{pmatrix} r & b \\ s & d \end{pmatrix} \), where:

- The first column is the constants of the two equations.
- The second column is the coefficients of the \( y \)'s in the two equations.

The denominator of the \( x \)-value is the determinant of the matrix \( \begin{pmatrix} a & c \\ b & d \end{pmatrix} \), which are the coefficients of the two equations we want to solve, in matrix form. So:

\[
x = \frac{D_x}{D}
\]

\[
y = \frac{as - rc}{ad - bc}
\]

From this we can see that the numerator of the \( y \)-value is the determinant of the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), where:

- The first column is the coefficients of the \( x \)'s in the two equations.
- The second column is the constants of the two equations.

The denominator of the \( y \)-value is the determinant of the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), which are the coefficients of the two equations we want to solve, in matrix form. So:

\[
y = \frac{D_y}{D}
\]

**Areas of difficulty and common mistakes**

- Students may find it difficult to multiply matrices. Let them use arrows as illustrated above and also let them write out their work.
- Students may find it difficult to solve simultaneous equations using matrices. Let them practise the method.

**Supplementary worked examples**

Solve these equations by using Cramer’s rule:

\[
\begin{align*}
3x - y &= 7 \\
-5x + 4y &= -2
\end{align*}
\]

**Solution**

**Step 1:** Write the equations in matrix form:

\[
\begin{pmatrix} 3 & -1 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}
\]

**Step 2:** Find the determinant of the first matrix:

\[
D = \begin{vmatrix} 3 & -1 \\ -5 & 4 \end{vmatrix} = (3)(4) - (-5)(-1) = 7
\]

**Step 3:** Now, take the matrix \( \begin{pmatrix} 3 & -1 \\ -5 & 4 \end{pmatrix} \) and replace the coefficients of \( x \) in the first column with the constants shown in the matrix \( \begin{pmatrix} 7 \\ -2 \end{pmatrix} \). The matrix will now look like this:

\[
\begin{pmatrix} 7 & -1 \\ -5 & 4 \end{pmatrix}
\]

**Step 4:** Calculate the determinant:

\[
D_x = \begin{vmatrix} 7 & -1 \\ -5 & 4 \end{vmatrix} = (7)(4) - (-1)(-5) = 26.
\]

**Step 5:** \( x = \frac{D_x}{D} = \frac{26}{7} \).

**Step 6:** Now, take the matrix \( \begin{pmatrix} 3 & -1 \\ -5 & 4 \end{pmatrix} \) and replace the coefficients of \( y \) in the second column with the constants shown in the matrix \( \begin{pmatrix} 7 \\ -2 \end{pmatrix} \). The matrix will now look like this:

\[
\begin{pmatrix} 3 & 7 \\ -5 & -2 \end{pmatrix}
\]

**Step 7:** Find the determinant:

\[
D_y = \begin{vmatrix} 3 & 7 \\ -5 & -2 \end{vmatrix} = (3)(-2) - (7)(-5) = 29
\]

**Step 8:** \( y = \frac{D_y}{D} = \frac{29}{7} \).
Learning objectives

By the end of this chapter, the students should be able to:

1. Recall and use various forms of the general equation of a straight line, including \( y = mx + c \) and \( px + qy + r = 0 \).
2. Recall and use various forms of the equations of lines that are parallel to the Cartesian axes.
3. Recall and use various expressions for the gradient of a straight line.
4. Find the intercepts that a line makes with the \( x \)- and \( y \)-axis and use them to sketch the line.
5. Find the distance between two points on a Cartesian plane.
6. Find the coordinates of the midpoint of a straight line joining two points of the Cartesian plane.
7. Recall and use the conditions for two lines to be a) parallel and b) perpendicular.
8. Calculate the angle between two lines on the Cartesian plane.

Teaching and learning materials

Students: Textbook, exercise book, writing materials, graph paper, mathematical instruments, tangent tables (see page 276 of the textbook) or scientific calculator.

Teacher: Graph board, chalkboard instruments, tangent tables or calculator. If an overhead projector is available: graph transparencies, and projector pens.

Glossary of terms

Cartesian plane is a flat surface in which the position of each point is defined by its distance along a horizontal axis (the \( x \)-axis) and its distance along a vertical axis (the \( y \)-axis) in this order.

- These axes intersect at the point \((0, 0)\) called the origin.
- The values on the \( x \)-axis to the left of the origin are negative and to the right they are positive.
- Above the origin the values on the \( y \)-axis are positive and below the origin they are negative.
- On this plane, each point can be found by its coordinates \((x, y)\).
- This plane is named after the mathematician René Descartes (1596–1650).

Teaching notes

- Explain to the students that the gradient of a line parallel to the \( x \)-axis \((y = p)\) is 0.
  - This is, because if we write the equation in the form \( y = mx + c \), where \( m \) represents the gradient, it would be \( y = 0 \times x + p \).
  - We can also say that in \( m = \frac{y_2 - y_1}{x_2 - x_1} \), the \( y \)-values do not change and \( y_2 - y_1 \) is, therefore, equal to 0.
  - So, \( m = \frac{0}{x_2 - x_1} = 0 \).
- The gradient of a line parallel to the \( y \)-axis, \( x = q \), is not defined.
  - In \( m = \frac{y_2 - y_1}{x_2 - x_1} \), the \( x \)-values do not change and \( x_2 - x_1 \) is, therefore, equal to 0.
  - So, \( m = \frac{0}{x_2 - x_1} \), which is undefined.
  - Remember that division by 0 is undefined because we say that, for example, \( \frac{10}{0} \) = 2 because \( 2 \times 5 = 10 \).
  - If, however, we say \( \frac{10}{0} = a \) number, that number \( \times \) by 0 must be equal to 10, which is impossible. So, division by 0 is undefined.
- Teach the students that one finds where the graph intersects the \( x \)-axis (\( x \)-intercepts) by making \( y = 0 \) in the equation of the graph. The reason we do this, is because on the \( x \)-axis all the \( y \)-coordinates are equal to 0.
  - One finds where the graph intersects the \( y \)-axis (\( y \)-intercept) by making \( x = 0 \) in the equation of the graph.
The reason we do this, is because on the y-axis all the x-coordinates are equal to 0.

- There are two methods to find the equation of a straight-line graph, if the gradient and another point are given.

Let us say we want to find the equation of a line that passes through the point \((-2, 3)\) and has a gradient of \(-3\).

**Method 1:**

If there is any point \((x, y)\) on the line the gradient between this point and \((-2, 3)\) and between the points \((-4, -1)\) and \((4, -5)\) are equal.

So, \[
\frac{y - 3}{x - (-2)} = \frac{-1 - (-5)}{-4 - (-2)}
\]

So, \[
y - 3 = -3(x + 2)
\]

\[
y = -3x - 6 + 3
\]

\[
y = -3x - 3
\]

**Method 2:**

First work out the gradient:

\[
m = \frac{-1 - (-5)}{-4 - (-2)} = \frac{-1 + 5}{-4 + 2} = \frac{4}{-2} = -2
\]

Now, you can write \(y = -2x + c\).

Substitute any one of the two points, for example, \((-4, -1)\):

\[
-1 = -2(-4) + c
\]

\[
-1 = 8 + c
\]

\[
c = -9
\]

\[
\therefore y = -2x - 3
\]

Explain to the students that they can choose to use any one of the two methods.

- Explain to the students that when we work out the midpoint of a line, we actually find the average between the two x-coordinates and the average between the two y-coordinates.

- When you want to calculate the angle between two lines, an alternate way to do this is to use the fact that the exterior angle of a triangle is equal to the sum of the two opposite interior angles.

So, Example 6 could also be done like this:

Then \(\angle BAC = \alpha - \beta\), because \(\alpha = \beta + \angle BAC\).
Areas of difficulty and common mistakes

* For some reason students tend to get the gradient wrong as \( \frac{x_2 - x_1}{y_2 - y_1} \). Tell them to check their work and to remember how the gradient is obtained.
* Sometimes students forget how to find the midpoint of a line. Tell them that, if we want to find the \( x \)-coordinate in the middle of two other \( x \)-coordinates, then we always add the two \( x \)-coordinates and divide the answer by 2. The same principle applies for finding the \( y \)-coordinate of the midpoint of a line segment.
* Students find it difficult to find the angle between two lines. Tell them to always first make a sketch of the two lines to find out what they can do to work out the angle between the two lines by using the fact that their gradients are the tan ratios of the angles the lines make with the \( x \)-axis.
* These angles are called the angles of inclination.
* Students sometimes also use the square root sign inappropriately when they work out the length of a line. For example:
  \[
  AB = \sqrt{64 + 36} = 8 + 6 = 14, \text{ instead of } \sqrt{100} = 10.
  \]
  Or, they write \( AB^2 = 64 + 36 = 100 = 10 \) instead of writing \( AB = \sqrt{64 + 36} = \sqrt{100} = 10 \).

\( \therefore AB = 10 \)

Supplementary worked examples

1. Determine the values of \( a \) and \( b \) when \( C \) is the midpoint of \( AB \), if:
   a) \( A(a, 4) \), \( B(3, b) \) and \( C(4, -6) \)
   b) \( A(a, -7) \), \( B(2a, b) \) and \( C(3, 4b) \)

2. For each of these write down the gradient of the line that is i) parallel to, and ii) perpendicular to, the given line:
   a) \( y = 3x + 2 \)
   b) \( 3x + 2y = 6 \)
   c) \[
   \]

3. Draw sketch graphs of:
   a) \( y = -2x \)
   b) \( y = -2 \)
   c) \( x = 2 \)
   d) \( y = \frac{1}{2}x \)

Solution

1. a) \[
  \frac{a + 3}{2} = 4
  \]
  \( \therefore a + 3 = 8 \)
  \( \therefore a = 5 \)
  \( \frac{4 + b}{2} = -6 \)
  \( \therefore 4 + b = -12 \)
  \( \therefore b = -16 \)

b) \[
  \frac{a + 2a}{2} = 3
  \]
  \( 3a = 6 \)
  \( \therefore a = 2 \)
  \( \frac{-7 + b}{2} = 4b \)
  \( -7 + b = 8b \)
  \( -7b = 7 \)
  \( \therefore b = -1 \)

2. a) i) \( 3 \) ii) \( \frac{-1}{3} \)
   b) First write the equation in gradient \( y \)-intercept form:
   \( 2y = -3x + 6 \) (add \(-3x\) to both sides)
   \( y = -\frac{3}{2}x + 3 \) (divide both sides by 2)
   i) \( -\frac{3}{2} \)
   ii) \( -\frac{3}{2} \times \frac{2}{3} = -1 \), so the answer is \( \frac{2}{3} \).
   c) The gradient is negative and equal to \(-\frac{5}{2}\).
   i) \( -\frac{5}{2} \)
   ii) \( -\frac{5}{2} \times \frac{2}{3} = -1 \), so the answer is \( \frac{2}{3} \).
   d) The gradient is positive and equal to 2.
   i) 2
   ii) \( 2 \times \left( -\frac{1}{2} \right) = -1 \), so the answer is \( -\frac{1}{2} \).
e) The gradient is positive and equal to $\frac{1}{4}$.
   i) $\frac{1}{4}$
   ii) $\frac{1}{4} \times (-4) = -1$, so the answer is $-4$.

3. a) 

When the $y$-intercept $= 0$, we cannot use the intercepts with the axes to draw the graph, because the $x$- and $y$-intercepts are equal and we need at least two points to draw the graph. So, we show the gradient and the fact that the line passes through the origin.

Note the direction of the line because of the fact that the gradient is negative.

b) 

You must show that the graph is parallel to the $x$-axis, because it is a sketch and facts are only facts if they are shown on a sketch.

c) 

You must show that the graph is parallel to the $y$-axis.

d) 

When the $y$-intercept $= 0$, we cannot use the intercepts with the axes to draw the graph, because the $x$- and $y$-intercepts are equal and we need at least two points to draw the graphs. So, we show the gradient and the fact that the line passes through the origin.

Note the direction of the line because of the fact that the gradient is positive.
Learning objectives

By the end of this chapter, the students should be able to:
1. Find the gradient functions of a simple quadratic curve and hence find the gradient of the curve at any point.
2. Apply the idea of limits to determine \( \frac{dy}{dx} \), the differential coefficient of \( y \) with respect to \( x \).
3. Differentiate polynomials using the rule that if \( y = Ax^n \), then \( \frac{dy}{dx} = Anx^{n-1} \).
4. Use the chain rule, and the product and quotient rules to differentiate complex algebraic expressions.
5. Apply differential calculus to determine:
   - rates of change including gradients, velocity and acceleration
   - turning points on a curve and curve sketching
   - solutions to real-life problems.

Teaching and learning materials

Teacher: Charts showing the basic rules of differentiation; calculus-related computer instructional materials where available.

Glossary of terms

Limit The notation ‘lim’ is called a limit and represents the value that something tends to become. In other words, the value comes so near to a certain value that we can predict what that value will be.

Gradient function is the derivative of a function and gives a formula to determine the gradient of the original function at any value of \( x \), or the gradient of the tangent to the original function at that \( x \). The gradient function or derivative can be expressed as \( f'(x) \), if the original function was written as \( f(x) = \ldots \); or \( \frac{dy}{dx} \), if the original function was written as \( y = \ldots \).

Teaching notes

- An alternate way of determining the gradient function \( f(x) = x^2 \) without using rules, for example, is:
  \[
  \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h
  \]

\[ f'(x) = \lim_{h \to 0} \frac{(2x + h) - 2x}{h} = \lim_{h \to 0} \frac{h}{h} = 2x. \]

It is very important that brackets are used, because we want the lim of \( 2x + h \) and not only the lim of \( 2x \).

- The derivative of a function can be understood as:
  - The gradient of the tangent at any value of \( x \) on the curve of the function.
  - The gradient of the function itself. (If the function is a curve, one can see the curve as consisting of infinitely small straight lines and the derivative is then the gradient of this small little bit of the curve.)
  - The instantaneous rate of change at any value of the independent variable (for example, \( x \) or \( t \)).
  - If \( f(x) = k \), then \( f'(x) = 0 \), because the gradient of \( y = k \) is equal to zero. The derivative of a constant is equal to 0.
  - If \( f(x) = mx + c \), then \( f'(x) = m \), because the gradient of \( y = mx + 3 \) is equal to \( m \).
• Explain to the students that when they determine the derivative using the rules of differentiation that they first have to:

- Expand expressions like \((3x - 4)^2\), 
  \((2x - 1)(2x + 1)\), 
  \((3x + 2)(4x - 5)\) and 
  \((x^2 + 1)(2x - 1)\) before applying the rules of differentiation.

- Write expressions such as \(\frac{5}{x^3} - \frac{3}{x^2} + \frac{2}{x}\) in the form \(5x^{-3} - 3x^{-2} + 2x^{-1}\) before applying the rules of differentiation. So:

  \[
  \frac{d}{dx}(5x^{-3} - 3x^{-2} + 2x^{-1}) = 5(-3)x^{-3-1} - 3(-2)x^{-2-1} + 2(-1)x^{-1-1} \\
  = -15x^{-4} + 6x^{-3} - 2x^{-2} \\
  = -\frac{15}{x^4} + \frac{6}{x^3} - \frac{2}{x^2}
  \]

  We always write the final answer with positive indices (exponents).

- Write expressions with root signs as expressions with fractional exponents, for example: \(\sqrt[3]{x^3} = x\). So:

  \[
  \frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{3}{4}(x^{3-1}) \\
  = \frac{3}{4}x^{-4} \\
  = \frac{3}{4x^4}
  \]

  Students would perhaps understand how to apply the chain rule; product and quotient rules better, if these rules are given in words:

  - **The chain rule in words:** Differentiate the outside function, then differentiate each function inside the brackets and multiply the results by each other.

  - **Product rule in words:** If you differentiate two factors, you differentiate the first factor and multiply it by the second factor, and then you differentiate the second factor and multiply it by the first factor. Then you add the two products.

  - **The quotient rule in words:** If you differentiate a quotient, you differentiate the part above the line (the numerator) and multiply the answer with the part underneath the line (the denominator). From this you subtract the derivative of the part underneath the line multiplied by the part above the line. This whole expression is divided by the square of the part underneath the line.

• Here is an alternate method for Example 15:

  \(y = x^2 - 4x + 3\)

  \(\frac{dy}{dx} = 2x - 4\)

  So, at \(x = 3\) the gradient of the curve = 2(3) - 4 = 2.

  Hence, we have \(y = 2x + c\).

  To find \(c\) we substitute the point (3, 1):

  \(1 = 2(3) + c\)

  \(\therefore c = -5\)

  Equation: \(y = 2x - 5\).

  If the \(y\)-coordinate of the point of contact of the tangent was not given, the \(x\)-coordinate of this point must be substituted into the equation of the original function to which the line is a tangent to find the corresponding value of \(y\) at the point. This point must then be substituted into \(y = 2x + c\) to find \(c\).

• The only way it is possible that the gradient of a tangent to a curve is equal to zero, is when the curve has a turning point at that \(x\)-coordinate. This turning point could either be a maximum turning point or minimum turning point.
If we want to know whether a turning point is a maximum or whether it is a minimum turning point, it is easier if tables like the ones below are used:

Example 17 could then be done like this:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient function = 4 − 2x</td>
<td>4</td>
<td>0</td>
<td>−2</td>
</tr>
<tr>
<td>Positive gradient</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Negative gradient</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 18 could be done like this:

<table>
<thead>
<tr>
<th>x</th>
<th>−2</th>
<th>−1</th>
<th>0</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>dy/(dx) = ((x - 2)(x + 1))</td>
<td>m = 4</td>
<td>m = 0</td>
<td>m = −2</td>
<td>m = 0</td>
<td>m = 4</td>
</tr>
<tr>
<td>m &gt; 0</td>
<td>m &lt; 0</td>
<td>m &gt; 0</td>
<td>m &lt; 0</td>
<td>m &gt; 0</td>
<td>m &lt; 0</td>
</tr>
</tbody>
</table>

• One could of course only substitute \(x = −1\) into \(y = 2x^3 − 3x^2 - 12x + 8\) to find the corresponding \(y\)-coordinate of the turning point and find, as shown, that the turning point is \((-1, 15)\).
• If the \(y\)-coordinate of the other turning point is calculated it is \(-12\). So, the turning point is \((2, -12)\).
• From these \(y\)-coordinates it can be seen which one of the turning points is a local maxima and which one is the local minima turning point.

We use the word “local” because of the fact that these points are not absolute maximum or minimum points, but are only a maxima or a minima for certain \(x\)-values.

We have a maximum turning point or a local maximum if the signs of the gradient function changes from positive to negative.

We have a minimum turning point or a local minimum if the signs of the gradient function changes from negative to positive.

Any point on the graph where the gradient of the tangent to the graph has a zero gradient, is called a stationary point.

If the signs of the gradient of the tangents to either side of the stationary point do not change, we call this specific stationary point a point of inflection.

If the signs of the gradients of the tangents to either side of the stationary point do not change, we call this specific stationary point a point of inflection.

Areas of difficulty and common mistakes

• When determining the limit of an expression with more than one term, the mistake is made to not put it in brackets. For example:
  \[ \lim_{h \to 0} [(2a + 4) + b] \] or \[ \lim_{h \to 0} (2a + b) \] and not \[ \lim_{h \to 0} (2a + 4) + b \] or \[ \lim_{h \to 0} 2a + b \].

• When determining the equation of the tangent to a graph and only the \(x\)-coordinate of the point of contact is given, students tend to forget that they have to substitute this \(x\)-value into the equation of the original function to find the corresponding \(y\)-coordinate of the point of contact and not into the gradient function. If this \(x\)-value is substituted into the gradient function, the gradient of the tangent is found, because the \(y\)-value of the gradient function is its value and, therefore, the gradient at that specific \(x\)-value.
Supplementary worked examples

Determine the equation of the tangent to the curve defined by $y = 2x\sqrt{17 - x^2}$, where $x = -1$.

Solution

$$\frac{dy}{dx} = 2\sqrt{17 - x^2} + \frac{1}{2}(17 - x^2)^{\frac{1}{2}}(-2x)(2x)$$

$$= 2\sqrt{17 - x^2} + \frac{(-2x)(2x)}{2\sqrt{17 - x^2}}$$

$$= 2\sqrt{17 - x^2} - \frac{2x^2}{\sqrt{17 - x^2}}$$  
(This is a combination of the chain rule and the product rule)

$m_{\text{tangent}} = 2\sqrt{17 - (-1)^2} - \frac{2(-1)^2}{\sqrt{17 - (-1)^2}}$

$$= 8 - \frac{1}{2}$$

$$= 7\frac{1}{2}$$

$y = 7\frac{1}{2}x + c$

Now substitute $x = -1$ into $y = 2x\sqrt{17 - x^2}$:

$y = 2(-1)\sqrt{17 - (-1)^2}$

$$= -2\sqrt{16}$$

$$= -8$$

Substitute $(-1, -8)$ into $y = 7\frac{1}{2}x + c$ to find $c$:

$-8 = 7\frac{1}{2}(-1) + c$

$c = -8 + 7\frac{1}{2}$

$$= \frac{1}{2}$$

$\therefore$ The equation of the tangent is: $y = 7\frac{1}{2}x - \frac{1}{2}$. 
Learning objectives
By the end of this chapter, the students should be able to:
1. Determine the equation of a line, or curve given its gradient function.
2. Use the inverse of differentiation to find the general solutions of simple differential equations.
3. Apply the rules of integration to integrate algebraic expressions.
4. Calculate the arbitrary constant, given sufficient relevant information.
5. Apply integral calculus to solve problems relating to:
   * coordinate geometry.
   * displacement, velocity, acceleration.
6. Evaluate definite integrals in the form $\int_a^b f(x)\,dx$.
7. Find the area under a curve by integrating between limits.

Teaching and learning materials
Teacher: Charts showing standard integrals; calculus-related computer instructional materials where available.

Teaching notes
* Integration can be seen as:
  - The inverse process of differentiation, and it is, therefore, also called anti-differentiation. It means finding the function that was differentiated to find the present function. This process is also called determining the indefinite integral.
  - The mathematical process by which the exact area under a curve, bounded by the x-axis, and for a certain interval is found. This process is called determining the definite integral.
* The rules for differentiation are:
  1. **Integral of a constant**: $\int k\,dx = kx + C$, where $k$ and $C$ are constants.
     *In words:* The integral of a constant is that constant times $x$, and then add $C$, the arbitrary constant of integration.
  2. **The integral of a power of a variable, for example $x^n$:** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$.
     *In words:* For a power of a variable (say $x$) add 1 to the exponent of $x$ and divide the power of $x$ by that same number and then add $C$, the arbitrary constant of integration.
   * If you want to find the integral of an expression like $(2x + 3)^2$, you must first expand it by determining the square of the expression. Then, you can determine the integral of each term separately.
   * You can check your answers by differentiating them to see if you get the function you had to integrate.
   * The velocity is the derivative of the function that represents displacement. So, to determine the function that represents displacement, the velocity function has to be integrated.
   * Acceleration is the derivative of the function that represents velocity. So, to determine the function that represents velocity, the acceleration function has to be integrated.
To determine the definite integral, we follow these steps:

**Step 1:** Use anti-differentiation to integrate the given function. (The integration constant, C, is not necessary.)

**Step 2:** Substitute the biggest \(x\)-value into the integral.

**Step 3:** Substitute the smallest \(x\)-value into the integral.

**Step 4:** Subtract the second answer from the first answer.

\[ \int_{a}^{b} f(x) \, dx \] is the general notation for the precise area under the curve. This is also called the definite integral with respect to \(x\).

- \(f(x)\) represents the heights of the rectangles for different \(x\)-values of the rectangles.
- \(dx\) represents the breadth of each rectangle.
- The elongated \(S\), \(\int\), represents the sum of all the infinitely thin rectangles \((dx \to 0)\). We also let the number of rectangles \((n)\) tend to \(\infty\) \((n \to \infty)\).
- The ‘\(a\)’ and ‘\(b\)’ represent the interval \([a, b]\) for the \(x\)-values.

**Areas of difficulty and common mistakes**

There are some difficult concepts to understand in this chapter.

They are, for example, the fact that rectangles can be infinitely thin and that we add an infinite number of these rectangles to find the area between a curve, the \(x\)-axis and from a certain \(x\)-value to a bigger \(x\)-value.

You will have to explain this very carefully.

**Supplementary worked examples**

Find the exact area under this curve (between the given \(x\)-values and the \(x\)-axis) by using the definite integral:

\[ y = -3(x - 2)(4 - x) \]

\[ a) \ x \in [0, 2] \]

\[ b) \ x \in [2, 4] \]

\[ c) \ x \in [0, 4] \]

**Solution**

a) \( \int_{0}^{2} (3x^3 - 18x^2 + 24x) \, dx \) (First remove the brackets)

\[ = \left( \frac{3x^4}{4} - 18 \frac{x^3}{3} + 24 \frac{x^2}{2} \right)_{0}^{2} \]

\[ = \left( \frac{3x^4}{4} - 6x^3 + 12x^2 \right)_{0}^{2} \]

\[ = \left( \frac{3}{4} (2)^4 - 6(2)^3 + 12(2)^2 \right) - 0 \]

\[ = 12 \text{ units}^2 \]

b) \( \int_{2}^{4} (3x^3 - 18x^2 + 24x) \, dx \)

\[ = \left( \frac{3x^4}{4} - 6x^3 + 12x^2 \right)_{2}^{4} \]

\[ = \left( \frac{3}{4} (4)^4 - 6(4)^3 + 12(4)^2 \right) - \left( \frac{3}{4} (2)^4 - 6(2)^3 + 12(2)^2 \right) \]

\[ = -144 - 12 \]

\[ = -156 \text{ units}^2 \]

\( \therefore \) Area = 156 units\(^2\) (Area cannot be negative)

c) Total area = (12 + 156) units\(^2\) = 178 units\(^2\)
Learning objectives
By the end of this chapter, the students should be able to:
1. Recall and use the method of calculating the standard deviation of a set of discrete data.
2. Interpret the variation or spread of a data set in terms of its standard deviation.
3. Calculate the mean deviation of a set of discrete data.
4. Select and use a working mean to simplify calculation.
5. Calculate the variance of a given data set.
6. Calculate the standard deviation of a set of grouped data.

Teaching and learning materials
Teacher: Newspaper articles, reports and data sets from recent health, market, population, weather and similar official and unofficial reports and studies.

Glossary of terms
Standard deviation shows the extent of the variation or spread of the distribution. The bigger the standard deviation, the wider is the spread of data (for example the Mathematics marks of a class). The smaller the standard deviation, the narrower the spread of data.

Teaching notes
* Standard deviation of data not organised in classes uses the actual values ($x$) to calculate the mean and the deviation from the mean.
* For grouped data, the values ($x$) are the class midpoint that is found by adding the lowest and highest values (the class limits) of an interval and dividing the answer by 2.
* When you calculate the mean, these steps must be followed:
  **Step 1** Multiply the frequency ($f$) that a value ($x$) occurs with the value ($x$) to get $fx$. If the data is organised in class intervals, you multiply the $f$ by the class midpoint ($x$) because the class midpoint represents all the values in that specific class interval.
  **Step 2** Add all the values you got in Step 1.
  **Step 3** Add all the frequencies.
  **Step 4** Divide the sum of $fx$ by the sum of $f$.
* When you calculate the variance, these steps must be followed:
  **Step 1** Add all the frequencies.
  **Step 2** Determine the difference ($d$) between the value ($x$) and the mean to get $d = x - \text{mean}$.
  **Step 3** Square all the differences ($d^2$) in Step 2 to remove the negative of some values.
  **Step 4** Multiply $f$ by the square of each of the differences to get $fd^2$ for each one.
  **Step 5** Add all the squares of differences multiplied by $f$, that is, add all the $fd^2$ of Step 4.
  **Step 6** Divide the total of the frequency multiplied by squares of the differences by the total of the frequencies. This gives the formula $\frac{\sum fd^2}{\sum f}$. This is called the variance.

The standard deviation is the square root of the variance $\sqrt{\frac{\sum fd^2}{\sum f}}$ and this formula for the standard deviation is the easiest to remember.
* The standard deviation must not be confused with the mean deviation which is calculated when the deviation is not squared and the positive values of the deviations are taken.
  **For data not organised in class intervals, you work this out by taking the positive value of the difference between the value ($x$) and the mean ($m$) to get $|x - m|$, where the vertical
lines indicate the size of $x - m$ (the sign is not taken into account). This is then divided by the total number of values ($n$). This gives the formula: $\frac{\sum|x - m|}{n}$.

- For grouped data you multiply the positive value of the difference between the class midpoint ($x$) and the mean ($m$) by the frequency ($f$) to get $f|x - m|$. All these values are then added and divided by total of the frequencies. All of this is then added to give the formula: $\frac{\sum|x - m|}{\sum f}$.

Areas of difficulty and common mistakes
This work is not really difficult if explained well. Students also do not make general mistakes.
### Learning objectives

By the end of this chapter, the students should be able to recall and know the following work:

1. Fractions, decimals, percentages and approximations.
2. Number bases.
3. Modular arithmetic.
4. Ratio and rate.
5. Proportion and mixtures.
7. Arithmetic and geometric progressions.
8. Matrices.

### Teaching and learning materials

**Students:** Textbook, an exercise book, writing materials and a calculator if possible.

**Teacher:** New General Mathematics 1, 2 and 3.

### Glossary of terms

**Elements** of a set are the members of the set. The symbol \( \in \) is used for element. The number of elements of a set, \( A \), for example, is called the cardinal number of \( A \) and written as \( n(A) \).

**Subset:** If \( A \) is a subset of \( B \), then \( A \) is part of \( B \).

\( A \) contains some or all of the elements of \( B \) and no elements that do not appear in \( B \). We write \( A \subseteq B \).

**Proper subset:** If \( A \) is a proper subset of \( B \), then \( A \) is a subset that consists of at least one, but not all, of the elements of \( B \). We write \( A \subset B \).

**Union of sets** The elements of the sets are put together without repeating any element. If \( A \) and \( B \) are united, we write \( A \cup B \). Then \( A \cup B \) will contain all the elements that are in \( A \) or \( B \).

**Intersection of sets:** The intersection of sets is the elements that these sets have in common. If the intersection of \( A \) and \( B \) is determined, we write \( A \cap B \) and this is the set that contains all the elements that are in \( A \) and \( B \).

**Disjoint:** If \( A \cap B = \emptyset \), (that is they have no elements in common) we say that sets \( A \) and \( B \) are disjoint.

**Finite set** is a set with a last element.

**Infinite set** is a set with an infinite number of elements and therefore has no last element.

### Teaching notes

**Fractions, decimals, percentages, approximations**

- Again remind the students that we can only add or subtract fractions if they are of the same kind. So we have to write them with the same denominator. For example:

\[
\frac{3}{5} + \frac{3}{4} = \frac{3 \times 4}{5 \times 4} + \frac{3 \times 5}{4 \times 5} = \frac{12}{20} + \frac{15}{20} = \frac{27}{20}
\]

(The LCM of 5 and 4 is 20. \( 5 \times 4 = 20 \) (and \( 4 \times 5 = 20 \)).

So, we multiply below and above the line by 4(5) and, therefore, do not change anything, because we multiplied by \( \frac{4}{4} = 1 \) and \( \frac{5}{5} = 1 \).

- Remind students to keep the order of operations in mind. (BODMAS means Brackets, Of, Division and Multiplication, Add and Subtract.)

- Remind the students why we multiply by the reciprocal of a fraction, if we divide by that fraction by doing this example:

Let’s say we have \( 0.66 \div \frac{2}{5} \).

We multiply by 0.66 by 3, because we want to know how many thirds there are in 0.66: \( 0.66 \times 3 = 1.98 \).

Then we divide this answer by 2, because we want to know how many 2’s there are in 1.98: \( 1.98 \div 2 = 0.99 \).

In one step:

\[
0.66 \div \frac{2}{5} = 0.66 \times \frac{3}{2} = 0.33 \times 3 = 0.99
\]
• Also remind students that, when decimal numbers with decimal fractions are multiplied, the answer has as many digits after the decimal point as there were in total. Let’s say we have $1.2 \times 0.06 \times 0.003 = 0.000216$ (the answer has 6 digits after the decimal point). The reason is that we can write:

$$\frac{12}{10} \times \frac{6}{100} \times \frac{3}{1000} = \frac{12 \times 6 \times 3}{10 \times 100 \times 1000} = \frac{216}{1,000,000} = 0.000216$$

• Again tell students that, if we want a certain percentage of a number or an amount, the percentage is always written as a 100th’s. So, $22\% \times 24 = \frac{22}{100} \times 24 = 5.28$

• Students can make their work much easier if they can remember these fractions for percentages or vice versa:

<table>
<thead>
<tr>
<th>Percentage</th>
<th>Equivalent Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>25%</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>50%</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>75%</td>
<td>$\frac{3}{4}$</td>
</tr>
<tr>
<td>12 1/2%</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>37 1/2%</td>
<td>$\frac{3}{8}$</td>
</tr>
<tr>
<td>62 1/2%</td>
<td>$\frac{5}{8}$</td>
</tr>
<tr>
<td>87 1/2%</td>
<td>$\frac{7}{8}$</td>
</tr>
<tr>
<td>33 1/3%</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>66 2/3%</td>
<td>$\frac{2}{3}$</td>
</tr>
</tbody>
</table>

Modular arithmetic

• If students do not understand adding and subtracting go back to a basic cycle with the correct number of steps and explain addition and subtraction by using the cycle.

  - Mod 7 means that a cycle has 7 steps for example and the principle is where in the cycle one would end.

  - In Example 15c): $4 \oplus 24$ means that you are on the 4 in the cycle and then you go 4 times around the cycle because $24 = 4 \times 6$.
    - Then you end at 4 again in the cycle.
    - The answer is, therefore, $4 \pmod{6}$.

  - In Example 15c): $5 \oplus 28$ means $5 \oplus 4 \oplus 24 = 3 \oplus 6 \oplus 4(6) = 3 \oplus 5(6)$.
    - This means that you start at 3 and go 5 times through the cycle and ends at 3 again.
    - The answer is, therefore, $3 \pmod{6}$.

Ratio and rate

• If students work with problems involving speed, time and distance, let them remember this triangle:

  Distance
  
  Speed
  
  Time

  This tells us that $\frac{\text{distance}}{\text{time}} = \text{speed}$, $\frac{\text{distance}}{\text{speed}} = \text{time}$ and that $\text{distance} = \text{speed} \times \text{time}$.

• Remind students that when comparing ratios, the ratio can either be expressed in the form $n : 1$, which comes down to expressing the ratio as a decimal fraction. Ratios can, however also be written as fractions with the same denominators by taking the LCM of the denominators. This method is easier to use to compare ratios if calculators are not available.

• When reducing a quantity in a certain ratio (Example 2), the same result is obtained if the quantity is multiplied by smallest number and divided by the biggest number in the ratio: $273 \, 000 \times \frac{11}{13}$. If the quantity is increased in a certain ratio, it is multiplied by the biggest number and divided by the smallest number in the ratio: $273 \, 000 \times \frac{13}{11}$.

  - In Example 19 one could also reason as follows: If 9 people take 21 days, 1 person takes $\frac{21 \times 9}{7}$ days (one person would work much longer). Then, 7 people would take $\frac{21 \times 9}{7}$ (7 people would take 7 times shorter).

Sets and applications

• A well-defined set clearly indicates what an element of the set is and what not.
  - A set identified as the set of all people taller than 2 m in Freetown is well defined. The reason is that its description cannot be interpreted differently.
  - On the other hand, if a set is identified as the set of tall people in Freetown, it is open to interpretation because different people could have different ideas of when a person is tall.

• Sets are equal if they have exactly the same elements. We use the symbol $=$. An empty set is always a subset of any set because a subset has no element that is not an element of any other set.

  • The number of subsets of any set is $2^n$. Exercise 5d) 4a).

• The number of proper subsets is $2^n – 1$ because the set itself must be left out.

• When students solve practical problems by using Venn diagrams, let them make the quantity they want to find, equal to $x$ and then let them fill in the remainder of the Venn diagrams in terms of $x$.

• If there is not a specific quantity that has to be
found (Example 9), let the students start with the quantity which is the intersection of all the sets and then complete the remainder of the Venn diagrams according to this value.

**Arithmetic and geometric progressions**

- Students do not always know when to use the formula $S_n = \frac{n}{2}(a + l)$ or when to use the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$ to determine the sum of an arithmetic series.
- Teach them that the first formula can only be used if they know the last term of the series. The second formula is used when the last term is not known.

When $r < 1$, the formula for the sum of a geometric series most convenient to use is $S_n = \frac{a(1 - r^n)}{1 - r}$.

When $r > 1$, the formula for the sum of a geometric series most convenient to use is $S_n = \frac{a(r^n - 1)}{r - 1}$.

- If an infinite geometric series converges, it means that its sum tends to a certain value. This can only happen if $-1 < r < 1$.

**Matrices**

Refer to Chapter 8 of this book.

**Areas of difficulty and common mistakes**

**Number bases**

- When students convert a number from the base 10 to another base, for example to the base 8, and they use reading the remainders upwards, they can in the end do this by rote without understanding what they are doing. Say, for example that 2 077 ten has to be converted to a number to the base 8. Then you can also explain it like this:

<table>
<thead>
<tr>
<th>8</th>
<th>2</th>
<th>0</th>
<th>7</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>259 + rem. 5</td>
<td>8</td>
<td>1 x 259 + 5 x 8^0</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>32 + rem. 3</td>
<td>8</td>
<td>2 x 32 + 3 x 8^1 + 5 x 8^0</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4 + rem. 0</td>
<td>8</td>
<td>3 x 4 + 0 x 8^2 + 3 x 8^1 + 5 x 8^0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Number to the base 8: 4035</td>
<td></td>
</tr>
</tbody>
</table>

- Remind students that they can only have the correct ratio, if the quantities that are compared are in the same unit.

**Ratio and rate**

Students do not take the word order into account when writing down a ratio. For example:

- The amount of N100 was divided between Anne and Gift in the ratio 2 : 3.
  - This means that Anne gets $\frac{2}{5} \times 100 = 40$ and Gift gets $\frac{3}{5} \times 100 = 60$.

**Proportion and mixtures**

Problems with mixtures and proportion could be difficult to solve and you should explain problems such as question 12 in Exercise 13e):

Let the number of sacks of rice bought at N6 200 be 5x, and the number of sacks of rice bought at N5 500 be 2x.

Total number of sacks of rice of both kinds of rice = 7x

Then the total price of the mixture

\[ = 5x(6 200) + 2x(5 500) \]

\[ = N42 000x \]

Selling price = $\frac{135}{106} \times 42 000x$

\[ = 56 700x \]

Selling price per sack = $\frac{56 700x}{7x} = N8 100$.
Learning objectives
By the end of this chapter, the students should be able to recall and know the following work:
1. Indices.
2. Logarithms (numbers > 1).
3. Logarithms (including numbers < 1).
4. Calculations without logarithms.
5. Theory of logarithms.

Teaching and learning materials
Students: Textbook, an exercise book, writing materials and a calculator if possible.
Teacher: New General Mathematics 1, 2 and 3.

Glossary of terms
Logarithm of a number is the exponent to which another fixed value, the base, must be raised to produce that number. For example, the logarithm of 1000 to base 10 is 3, because 10 to the power 3 is 1000: 1000 = 10 × 10 × 10 = 10^3.

Antilogarithm is the number of which a given number is the logarithm. For example, the antilogarithm of 3, if the base was 10 is 1000.

Evaluate means to determine the number value of something.

Teaching notes
Indices
* You could explain the negative exponent like this: \( \frac{2^3}{2^5} = 2^{3-5} = 2^{-2} \).

Usually, when we divide, we subtract the exponents of the equal bases where the biggest exponent is: \( \frac{2^7}{2^5} = \frac{1}{2^{7-5}} = \frac{1}{2^2} \).

From this we can deduce that \( 2^{-2} = \frac{1}{2^2} \).

Or \( \frac{2^5}{2^2} = 2^{5-2} = 2^3 \).

But if we forget that we always subtract exponents of equal bases where the biggest exponent is, the calculation can be done like this: \( \frac{2^7}{2^5} = \frac{1}{2^{7-5}} = \frac{1}{2^2} \).

So, \( 2^3 = \frac{1}{2^{-3}} \).

Therefore, to write numbers with positive indices, we write the power of the base with a negative exponent, on the opposite side of the division line, for example, \( \frac{1}{x^3} = \frac{x^3}{1} \) or \( x^{-3} = \frac{1}{x^3} \).

* You could explain why \( x \neq 0 \) in \( x^0 = 1 \) like this:
If \( x = 0 \), we may have that \( x^0 \) resulted from \( \frac{0}{0} \).

Here we divided by 0 which is not defined.

Then you can explain why division by 0 is not defined, like this:
Let’s say we take \( \frac{8}{2} = 4 \). This is because \( 2 \times 4 = 8 \).

Also \( \frac{0}{2} = 0 \), because \( 2 \times 0 = 0 \).

Now, if we take \( \frac{8}{0} \) is any number, then that number \( \times 0 \) must be equal to 8. That, however, is impossible, because there is no number that we can multiply by 0 that will give 8.

So, division by zero is not defined.

Logarithms (numbers > 1)
* Remind students what the word logarithm really means.
  - Emphasise the following: If \( 10^{2.301} = 200 \), then \( \log_{10} 200 = 2.301 \).
  - In words: log base 10 of 200 is the exponent to which 10 must be raised to give 200.

* Also remind students what antilog means. If, for example \( 10^{2.301} = 200 \), the antilog means that we want to know what the answer of \( 10^{2.301} \) is.

* Remind students why they add logarithms of numbers if they multiply the numbers, and why they subtract logarithms of numbers if they divide these numbers by each other.
• Emphasise that logarithms are exponents and that the first two exponential laws are:

- **Law I**
  \[ a^x \times b^y = a^{x+y} \]. For example,
  \[ a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) \]
  \[ = a \times a \times a \times a \times a \times a \times a \times a \]
  \[ = a^7 \]

- **Law II**
  \[ a^x + a^y = a^{x-y} \text{, where } x > y \]. For example, \( \frac{a^5}{a^2} = a^{5-2} = a^3 \).

- So, since logarithms to the base 10 are the exponents of 10:
  - We add the logs of the numbers, if we multiply the numbers.
  - We subtract the logs of the numbers, if we divide the numbers.

• Remind students that, when working out a number to a certain power, we multiply the log of the number with the power. Again emphasise this exponential law since logs to the base 10 are the same as the exponents of 10 that will give the number:
  \[ (a^m)^n = a^{mn} \]. For example, \( (a^4)^2 \)
  \[ = (a \times a \times a \times a) \times (a \times a \times a \times a) \]
  \[ = a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \]
  \[ = a^6 = a^{4 \times 2} = a^8 \] and \[ \sqrt[6]{a^3} \]
  \[ = (a^{3})^{\frac{1}{2}} = a^{\frac{3}{2}} = a^{\frac{6}{2}} = a^{3} \).

**Calculations without logarithms**

Here students must use their knowledge of decimal numbers, square roots and exponents to avoid using tables of squares, square roots, logarithms or anti-logarithms. You could remind them of the following:

• When dividing with a number a decimal fraction part, make this number a whole number by multiplying by the same number below and above the line.

• Write numbers with decimal parts as common fractions, 1.44, for example, becomes \( \frac{144}{100} \). This is especially useful if you want to get the square root of 1.44.

• Know the exponential laws and apply them as shown above. For example, if you want to evaluate \( \sqrt{\frac{8.1 \times 10^3}{1.44 \times 10^2}} \). Then you do the following:
  \[ \sqrt{\frac{81 \times 10^{-1} \times 10^3}{144 \times 10^2}} = \sqrt{\frac{81 \times 10^{-1}}{144 \times 10^2 \times 10^2}} \]
  \[ = \sqrt{\frac{9 \times 10^{-2}}{12 \times 10^2}} = \frac{3 \times 10^{-2}}{4 \times 10^2} = \frac{3}{4000} \]
Learning objectives
By the end of this chapter, the students should be able to recall and know the following work:
1. Simplification.
2. Factorisation.
3. Algebraic fractions.
4. Substitution.
5. Variation.
6. Calculus.

Teaching and learning materials
Students: Textbook, an exercise book, writing materials and a calculator if possible.
Teacher: New General Mathematics 1, 2 and 3.

Teaching notes
Simplification
• Again emphasise that simplifying an algebraic expression is not merely a game with certain rules.
  It means that an algebraic expression is written in a different form. This other form is simpler or easier to use in certain other situations.
  For example, if you want to solve a quadratic equation or simplify an algebraic fraction, the factorised form of such an expression is a simpler form for this purpose and that is one of the reasons that we learn how to factorise algebraic expressions.
• When we say that “like terms can be added” and “unlike terms cannot be added”, refrain from using words like “apples” for $a$ and “bananas” for $b$ or the name of any other object for letter symbols.
  It is absolutely wrong to reason like this, because these letter symbols are variables, which means that they represent any number and not objects.
  It would be correct, however, if you say that we do not know whether $a$ stands for 2 and $b$ for 3.
  As it is written here it is impossible to write the expression $3a + 2b$ in another form.

This expression simply is a computing procedure which says: “three time a certain number added to 2 times another number and cannot be written in any other way which will make the procedure simpler for substitution”, for example.
• When brackets are removed, emphasise that students multiply each term inside the bracket by the number or expression directly in front of the bracket (there is no $+$ or $-$ sign between the number and the bracket) and that the sign of this number is included when the bracket is removed by multiplication. Brackets are also removed from the inside brackets to the outside brackets.
• Expansion of brackets means that each term in the one bracket is multiplied by each term in the other bracket. This again is writing an algebraic expression in another form.
  • Sometimes the factorised form of an expression is its simplest form especially if we want to find a common denominator.
  • Sometimes the expanded form is the simpler form, especially if we want to find where graph intersects the $y$-axis.

Factorisation
• Before any kind of factorisation is done, first take out the common factor. The common factor is the highest number that will divide into each term of the expression or the lowest power of the letter symbol that occurs in all the terms.
When you have to group terms together to find a common factor which is a bracket, do not put brackets around the terms from which you want to take out a common factor, because the + or − signs in the brackets may then be wrong. Emphasise that, if the negative of the common factor is taken out, the signs of the terms in brackets also change to the opposite signs.

When taking out −1 to change signs, remember that:
- \((x - y)^2 = (y - x)^2\)
  - For example, let \(x = 5\) and \(y = 2\):
    - \((5 - 2)^2 = (3)^2 = 9\) and \((2 - 5)^2 = (-3)^2 = 9\).
  - \(a + b = -(a - b)\)
    - For example, let \(a = -7\) and \(b = 5\):
      - \(-7 + 5 = -2\) and \(-(7 - 5) = -(2) = -2\)
  - \(-a - b = -(a + b)\)
    - For example, let \(a = 10\) and \(b = 7\):
      - \(-10 - 7 = -17\) and \(-(10 + 7) = -(17) = -17\)

When factorising quadratic expressions the students must be reminded that:
- If the sign of the last term is positive, the two brackets have the same sign, namely the sign in front of the middle term:
  - \(x^2 + bx + c\) (both brackets have positive signs)
  - \(x^2 - bx + c\) (both brackets have negative signs)

- If the sign of the last term is negative, the two brackets have opposite signs. The biggest product will have the sign of the middle term:
  - \(x^2 + bx - c\) (the biggest product will be positive)
  - \(x^2 - bx - c\) (the biggest product will be negative)

**Substitution**
- To avoid errors, insist that students use brackets.
- Let them immediately write brackets instead of the letter symbols and then write the numeric values of the letters into the brackets.
- Let students develop this habit even if brackets are not necessary.

**Variation**
Remind students again of the following:
- If \(x\) is the independent variable and \(y\) is the dependent variable, they vary directly if there is a constant number \(k \neq 0\) such that \(y = kx\). The number \(k\) is the constant of variation. If two quantities vary directly, we say that they have a direct variation.
  - This means that, if \(x\) is multiplied by a number, then \(y\) is also multiplied by that same number.
  - It also means that, if the value of one quantity increases, the value of the other quantity increases in the same ratio.
- The variables \(x\) and \(y\) vary inversely, if for a constant \(k \neq 0\), \(y = \frac{k}{x}\) or \(xy = k\). The number \(k\) is the constant of variation and we say that the two quantities have an inverse variation.
  - This also means that, if a value of the independent variable \((x)\) is multiplied by 2, for example, then the corresponding value of \(y\) is multiplied by the multiplicative inverse of 2, namely, \(\frac{1}{2}\). So, if the value of one of the variables is multiplied by a number, the value of the other variable is multiplied by the multiplicative inverse of that number.
  - If two variables vary inversely, it means that, if the value of one increases, the value of the other decreases in the same ratio or increases inversely.

When we say \(c\) varies jointly to a set of variables, it means that \(c\) varies directly and/or inversely to each variable one at a time. If \(c\) varies directly to \(a\) and inversely to \(b\), the equation will be of the form \(c = \frac{ka}{b}\), where \(k\) is the constant of variation and \(k \neq 0\).

The area \((A)\) of a triangle, for example, varies directly to the length of its base \((b)\) and to the length \((b)\) of its perpendicular height. The equation, therefore, is \(A = kbh\) and \(k = \frac{1}{2}\). So, the final equation is \(A = \frac{1}{2}bh\).
When $L$ varies partially to $F$, then $L$, is the sum of a constant number and a constant multiple of $F$. This is called partial variation. The formula is then of the form $L = kF + c$ where $k$ and $c$ are constants.

We can also say that $k$ is the constant of variation and $c$ is the initial value of $L$.

We say $L$ varies partially to $F$ because in the equation $L = kF$, $L$ varies directly to $F$, but now something is added.

If a force is applied to a spring, we know that the increase in length varies directly to the force applied to the spring and if we want to get the length of the spring, we have to add the increase in length of the spring to its original length.

**Calculus**

See Chapter 10 of this book.

**Areas of difficulty and common mistakes**

**Simplification**

When removing brackets by multiplication, it should be remembered that one multiplies all the terms inside the brackets with the number and its negative sign, for example.

- If there is only a negative sign in front of the brackets, multiply all the terms by $-1$.

For example:

\[
3 - a(a - 5 - 6a) = 3 + a(-1)(5) + (-1)(-6a)) = 3 - a(a - 5 + 6a) = 3 - a(-a) - 5(-a) + (-a)(-6a) = 3 - a^2 + 5a + 6a^2 = 3 + 5a + 5a^2
\]

- $(2x - 5y)^2 \neq 4x^2 + 25y^2$. To avoid this mistake:

We always write: $(2x - 5y)^2$ as $(2x - 5y)(2x - 5y)$. Then, multiply each term in the first bracket with each term in the second bracket.

**Factorisation**

When grouping terms to take out a common factor of each pair of terms, it is wise not to use brackets to show which two terms you are grouping together.

You can make mistakes with signs, for example, $cd - ce - d^2 + de \neq (cd - ce) - (d^2 + de)$. The last part not correct, because $(-1)(de) = -de$, and in the original expression we had a positive "de".

To avoid this kind of mistake, brackets should only be used when the common factor is actually taken out:

\[
c(d - e) - d(d - e) = (d - e)(c - d).
\]

**Algebraic fractions**

The main mistake students make when simplifying fractions, is that they cancel terms.

Insist that they first factorise to avoid mistakes like the following: $\frac{9x^2 - 4y^2}{3x - 2y} = 3x + 2y$.

Notice that the answer is correct, but students could have reasoned that $\frac{9x^2}{3x} = 3x$ and $\frac{-4y^2}{-2y} = +2y$.

If you want to make sure that students did not reason like this, you must insist that they show the following in their working:

\[
\frac{9x^2 - 4y^2}{3x - 2y} = \frac{(3x + 2y)(3x - 2y)}{3x - 2y} = 3x + 2y
\]
Learning objectives
By the end of this chapter, the students should be able to recall and know the following work:
1. Linear equations.
2. Change of the subject of the formula.
3. Linear inequalities.
4. Simultaneous linear equations.
5. Quadratic equations.
7. Word problems.

Teaching and learning materials
Students: Textbook, an exercise book, writing materials and a calculator if possible.
Teacher: New General Mathematics 1, 2 and 3.

Teaching notes
Linear equations
• Again explain to the class that, if we solve an equation, we are looking for the value of the variable that will make the algebraic sentence or equation true. If, for example, $3 - 2x = 7$, we can say that $3 - (-4) = 7$. So, $2x = -4 \therefore x = -2$.
• It is very important that the balance method is again consciously taught as shown in the textbook.

Do not tell the students that terms are “taken over”.
An equal sign does not work like magic to change all the signs to opposite signs on the other side of the equal sign. There is a logical reason why the signs change.
If students do not understand this, they may later experience problems when working with letter symbols only when they change the subject of a formula.
• When we solve an equation such as $\frac{1}{x} + \frac{4}{3x} - 1 = 0$, emphasise that $x \neq 0$, because division by 0 is undefined.

• These steps are followed when solving equations:
  Step 1: Clear all fractions by multiplying each term (even if it is not necessary) both sides of the equation, by the LCM of the denominators of the fraction.
  Step 2: Remove all brackets by multiplication.
  Step 3: Add all the like terms on both sides of the equation.
  Step 4: Add a term with a sign opposite to the term that you want on the other side of the equation. (You always want the terms with the variable on the left-hand side and all the other terms on the right-hand side of the equal sign.)
  Step 5: Add all the like terms on both sides of the equation.
  Step 6: Multiply both sides by the reciprocal of the coefficient of the unknown.

Change the subject of the formula
• These steps can be followed if we want to change the subject of the formula:
  Step 1: Clear fractions by multiplying by the LCM of the denominators.
  Step 2: If there are square roots, square both sides of the equation.
  Step 3: Remove brackets by multiplication.
  Step 4: Add and subtract the same quantities both sides of the formula to get the subject of the formula to the left-hand side of the formula.
Step 5: Multiply both sides of the formula by the reciprocal of the coefficient of the subject to get 1 x the subject of the formula.

Step 6: If more than one term has the letter that you want to make the subject of the formula, take the letter out as a common factor and then divide both sides of the equation by the bracket.

Here students must be very sure of how to use the balance method. If they are used to “taking something over”, they will now make the most terrible mistakes, because they will become totally confused when faced with letter symbols only.

Linear inequalities
Tell students again that an inequality is solved just like an equation except when we multiply or divide by negative numbers. Then the inequality sign is reversed.

Simultaneous linear equations
- Before you do example 10 with the class tell them the following:
  When we have a linear equation in two variables (x and y), for example, there are an infinite number of values of x and y that will make it true.
  If we have two linear equations in two variables (x and y), for example, there is only one value of x and one value of y that will make them true at the same time.
  Each one of these equations can be represented by a straight-line graph.
  The point where these two lines intersect represents the value of x and of y that will make them true simultaneously. This can only happen if the two straight-line graphs are not parallel.
- Again make students aware of the following:
  Number the original equations and the equations that are created and using the numbers of the equations, say what is done to the equations, for example, 2 x 3 or 3 – 6, and so on. That way there could be no confusion.
- In order to eliminate variables:
  Subtract equations, if the coefficients of the same variable are equal but have opposite signs. For example:
  -2x – 3y = 10 … 1
  2x + 5y = –14 … 2
  1 – 2: –8y = 24
  ∴ y = –3
- Add equations, if the coefficients of the same variable are equal but have opposite signs. For example:
  -2x – 3y = 10 … 1
  2x + 5y = –14 … 2
  1 + 2: 2y = –4
  ∴ y = –2
- When subtracting two equations, students sometimes forget that the signs of the equation at the bottom change.
  - Again emphasise that when subtracting 2 from 1 above, the left-hand side can also be written as
    2x – 3y – (2x + 5y) = 2x – 3y – 2x – 5y = –8y.
  - The right-hand side can be written as
- In equations such as 3x – 2y = 24 and 4x – 9y = 36, students must realise that to eliminate x, they must first get the LCM of 3 and 4 which is 12.
  Then the first equation must be multiplied by 4 because 4 x 3 = 12 and the second equation must be multiplied by 3 because 3 x 4 = 12:
    3x – 2y = 24 … 1
    4x – 9y = 36 … 2
  1 x 4: 12x – 8y = 96 … 3
  2 x 3: 12x – 27y = 108 … 4
  They could of course have used the LCM of 2 and 9, which is 18 and multiplied the first equation by 9 and the second by 2 to eliminate y.

Quadratic equations
- When solving quadratic equations by means of factorisation, the zero product principle is applied. It states that, if A x B = 0, then A = 0 or B = 0 or both A and B are equal to 0. So, the left-hand side of a quadratic equation must always be in factor form and the right-hand side must be equal to zero.
- Emphasise that students follow these steps when they have to solve (2x + 1)(x – 1) = 12, for example:
  Step 1: Subtract 12 from both sides of the equation to get the RHS = 0:
    (2x + 3)(x – 1) = 12 = 0
  Step 2: Remove the brackets by multiplication:
    2x{\textsuperscript{2}} + x – 3 = 12 = 0
  Step 3: Add like terms:
    2x{\textsuperscript{2}} + x – 15 = 0
  Step 4: Factorise
    (2x – 5)(x + 3) = 0
Step 5: Use the Zero product principle to solve for \( x \):
\[
2x - 5 = 0 \quad \text{or} \quad x + 3 = 0
\]
\[
x = \frac{5}{2} \quad \text{or} \quad x = -3.
\]
* Students should apply these principles when trying to solve a quadratic equation:
  1. First try to factorise the equation
  2. If you cannot factorise the equation (even if it does have factors), use the quadratic formula to solve it.
  3. You only use the completion of the square to solve a quadratic equation when you are specifically asked to use this method.
* When students use the quadratic formula. Stress that they follow these steps:
  1. Rewrite the equation in the standard form of \( ax^2 + bx + c = 0 \).
  2. Write down the values of \( a \), \( b \) and \( c \) as \( a = \ldots, b = \ldots \) and \( c = \ldots \).
  3. Write down the quadratic formula:
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
  4. Then write brackets where the letters were: \( x = \frac{-(\ldots) \pm \sqrt{(\ldots)^2 - 4(\ldots)(\ldots)}}{2(\ldots)} \).
  5. Write the values of \( a \), \( b \) and \( c \) into the brackets.
  6. Use a calculator to work out the answer.

**Simultaneous linear and quadratic equations**

* When solving simultaneous linear and quadratic equations, students tend to make matters very difficult for themselves by choosing to make the variable, which will result in them working with a fraction, the subject of the linear equation. For example:
  If \( 3x + y = 10 \), solve for \( y \) and not \( x \).
  If we solve for \( y \), then \( y = 10 - 3x \).
  On the other hand, if we solve for \( x \), then \( x = \frac{10 - y}{3} \).
  So, if the value of \( x \) has to substituted into a quadratic equation, the work will be much more difficult.
  So, insist that students always ask themselves what the easiest option is and then make that variable the subject of the formula.
* Sometimes students find it difficult to factorise a quadratic equation. Teach them to use the quadratic formula if they find it difficult to get the factors.

**Word problems**

These problems are always difficult for the students. They could try the following to make the work easier:

* Let the quantity(-ies) you want to find be equal to a letter symbol(s).
* Take the problem sentence by sentence and write the facts in terms of your letter symbol(s).
* If an area or perimeter is involved, make a drawing and write the length and breadth in terms of your letter symbol(s).
* Organise the facts in a table. Example 21 can for example be done like this:

<table>
<thead>
<tr>
<th>Price</th>
<th>Number of packets</th>
<th>Price/packet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Old</td>
<td>( n )</td>
<td>( \frac{2,160}{n} )</td>
</tr>
<tr>
<td>New</td>
<td>( n + 3 )</td>
<td>( \frac{2,160}{n+3} )</td>
</tr>
</tbody>
</table>

\[
\frac{2\,160}{n} - \frac{2\,160}{n+3} = 24
\]

\[
(n + 3)2\,160 - 2\,160n = 24n(n + 3)
\]

\[
24n^2 + 72n - 6\,480 = 0
\]

\[
n^2 + 3n - 270 = 0, \text{ and so on.}
\]

**Areas of difficulty and common mistakes**

**Linear equations**

* When removing brackets as in \( 3(4c - 7) - (4c - 1) = 0 \), students tend to forget to multiply the second term in the bracket, also by the \(-1\) in front of the bracket. Emphasise that which is directly in front of the brackets, applies to all the terms inside the bracket.
* When solving equations with fractions where the numerator consists of more than one term, you must emphasise that this step must not be skipped:
\[
15\left(\frac{2x + 2}{3}\right) - 15\left(\frac{9x - 2}{5}\right) = 2(15)
\]
\[
5(7x + 2) - 3(9x - 2) = 30 \quad \text{(do not leave this step out)}
\]
The reason is that in the second term (in this example) the student then multiplies just $9x$ by $-3$ and $-2$ either stays the same or they write $-6$ instead of $+6$, because $-2$ is multiplied by only $3$ and not $-3$.

- When there is a whole number and both sides of the equations is multiplied by the LCM of the denominators, some students forget that this whole number (although it is not a fraction) must also be multiplied by the LCM of the denominators. Emphasise that each term both sides of the equation must be multiplied by the LCM of the denominators. Each term is treated the same.

**Change the subject of the formula**

When students are asked to change the subject of the formula $P = a W + B$ to $W$, for example, they sometimes tend to do the following:

$$a W = P - B$$

$$\therefore W = P - B - a.$$  

They were “taking over” without knowing what they were doing.  
If this happens, go back and again explain how the balance method works.

**Linear inequalities**

Students tend to write the inequality shown in the sketch: as $-2 > x > 4$. (You read it as: “$x$ is smaller than $-2$ and bigger than $4$”.)  

There is no number that is smaller than $-2$ and bigger than $4$.

Teach students that there are two parts and that the above inequality is actually the result of the union of two inequalities: $\{x; x < -2, x \in \mathbb{R}\} \cup \{x; x > 4, x \in \mathbb{R}\}$ and that they should always write: $x < -2 \textbf{ or } x > 4$. The word “or” represents the “union” of two sets.

**Simultaneous linear equations**

Students make mistakes if they substitute the value of the variable they found into an equation to find the value of the other variable.  
The main reason for these errors is the fact that they do not use brackets for substitution.  
Let us say that they found that $x$, for example, is equal to $-3$.  
They have to substitute the value of $x$ into $-3x + 2y = -4$ to find the value of $y$.  
Then, to prevent errors, they should substitute the value of $x$ like this: $-3(-3) + 2y = -4$.  

**Learning objectives**

By the end of this chapter, the students should be able to recall and know the following work:

1. Linear graphs.
2. Linear programming.
3. Quadratic graphs.
4. Sketch graphs.
5. Inverse functions
6. Relations and functions.

**Teaching and learning materials**

**Students:** Textbook, an exercise book, writing materials and a calculator if possible.

**Teacher:** New General Mathematics 1, 2 and 3.

**Teaching notes**

**Linear graphs**

- If you need to draw the graph of a linear function, there are several methods you could use:
  - You could complete a table of at least three values.
    - Always choose a negative number, 0 and positive number.
    - Choose the values in such a way that you do not get fractions.
  - Example:

    | $x$ | $y = -\frac{1}{2}x + 1$ |
    |-----|-----------------------|
    | -2  | 2                     |
    | 0   | 1                     |
    | 2   | 0                     |

  In this example, -2 and 2 were chosen so that there could be no fractions. (Fractions make plotting the points difficult).
  - You then plot the points in the table on a Cartesian plane and draw a line through them. If they do not lie in a line, you made a mistake in the table.
  - You could work out the $x$-intercept (where the graph intersects the $x$-axis) and the $y$-intercept (where the graph intersects the $y$-axis).
  - On the $x$-axis all the $y$-coordinates are equal to 0. So, to work out the $x$-intercept, you make $y = 0$.
  - On the $y$-axis all the $x$-coordinates are equal to 0. So, to work out the $y$-intercept, you make $x = 0$.

- For example, $y = -\frac{1}{2}x + 1$:
  - $x$-intercept:
    \[ -\frac{1}{2}x + 1 = 0 \]
    \[ -\frac{1}{2}x = -1 \]
    \[ \therefore x = 2 \]
  - $y$-intercept:
    \[ y = -\frac{1}{2}(0) + 1 \]
    \[ \therefore y = 1 \]

  To draw the graph, you plot the $x$- and $y$-intercepts and draw a line through them.

  The drawback of this method is that you could have made a mistake with one or both of the intercepts and still be able to draw a straight line through the two points. (You can draw a straight line through any two points).

- You can use the gradient and one other point:

  - For example, sketch the line that passes through the point $(5, -2)$ and has a gradient of $-\frac{4}{3}$.
  - These steps can be followed:
    a) First plot the point
    b) For the gradient $= -\frac{4}{3}$:
      - From the point go 4 units down and 3 units to the right.
    c) Or for the gradient $= \frac{4}{3}$:
      - From the point go 3 units to the left and 4 units up.

  **Always measure the gradient from the given point** (which could also be a $y$-intercept or an $x$-intercept).
Remember for the change in the $y$-values or the vertical change:
- Upwards is positive ($\uparrow$).
- Downwards is negative ($\downarrow$).

Remember for the change in $x$-values or the horizontal change:
- To the right is positive ($\rightarrow$).
- To the left is negative ($\leftarrow$).

* If you want to find the equation of a straight-line graph you can also use these two methods. Other methods are discussed in Chapter 16 of NGM Book 2:

1. Determine the equation of a straight line with gradient $-\frac{1}{3}$ that passes through the point $(-3, 2)$. Use the equation $y = mx + c$ for the straight line, where $m$ represents the gradient and $c$ represents the $y$-intercept.

   **Step 1:** Since $m = -\frac{1}{3}$, we can write $y = -\frac{1}{3}x + c$.

   **Step 2:** To find $c$, we substitute the given point $(-3, 2)$ into $y = -\frac{1}{3}x + c$.

   $2 = -\frac{1}{3}(-3) + c$
   $2 = 1 + c$

   $\therefore c = 1$

   **Step 3:** The answer is: $y = -\frac{1}{3}x + 1$.

2. Find the equation of the straight line that passes through the points $(1, 4)$ and $(-2, 6)$.

   **Step 1:** Determine the gradient:

   $m = \frac{6 - 4}{-2 - 1}$
   $= \frac{2}{-3}$
   $= -\frac{2}{3}$

   or

   $m = \frac{4 - 6}{1 - (-2)}$
   $= \frac{2}{3}$
   $= \frac{2}{3}$

   (Note that it does not matter which $y$-value you take first, as long as you take the corresponding $x$-value also first.)

   **Step 2:** Now the equation so far is: $y = -\frac{2}{3}x + c$.

   **Step 3:** Substitute any of the two given points into $y = -\frac{2}{3}x + c$.

   - If we use the point $(1, 4)$:
     $4 = -\frac{2}{3}(1) + c$
     $\therefore c = 4 + \frac{2}{3}$
     $= \frac{14}{3}$

   - If we use the point $(-2, 6)$:
     $6 = -\frac{2}{3}(-2) + c$
     $= \frac{4}{3} + c$
     $= \frac{14}{3}$
     $\therefore c = 6 - \frac{14}{3}$
     $= \frac{2}{3}$

   **Step 4:** The answer is $y = -\frac{2}{3}x + \frac{2}{3}$.

* When $y$ is the subject of the formula, the region above the line satisfies the inequality $y > mx + c$ and the region below the line satisfies the inequality $y < mx + c$. If you have $\leq$ or $\geq$, the line is solid, because it is included. If the inequality is only $<$ or $>$, the line is not included and is then shown as a broken line.

* Students need to know that, if an inequality is represented by means of a solid line, it means that the numbers represented are Real numbers.

* If the numbers are
  - Natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$ or
  - Counting numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ or
  - Integers, $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, \ldots\}$, then separate numbers as dots must be shown on the number line.

### Linear programming

In linear programming certain words are used to indicate inequality. Here is a summary of possibilities:

* At least means at the minimum or not less than or bigger than or equal $\geq$
* Not more than means less than or equal $\leq$
* Up to means at the most or the maximum or less than or equal $\leq$
* The minimum means bigger than or equal $\geq$
* Not less than means it must be more than or equal $\geq$
Quadratic graphs

- When you want to solve the equation \( ax^2 + bx + c = 0 \) graphically, you read off the values where the graph of \( y = ax^2 + bx + c \) intersects the \( x \)-axis, because on the \( x \)-axis all the values of \( y \) are 0.

If the quadratic equation has:
- Two roots, the graph of \( y = ax^2 + bx + c \) intersects the \( x \)-axis in two places.
- One root or can be written as a perfect square \((\ldots)^2\), the graph \( y = ax^2 + bx + c \) intersects the \( x \)-axis in only one point or just touches the \( x \)-axis.
- No real roots, the graph of \( y = ax^2 + bx + c \) does not intersect the \( x \)-axis at all.

- When you want to solve simultaneous equations such as \( mx + c = ax^2 + bx + c \) graphically, you draw precise graphs on graph paper of \( y = mx + c \) and \( y = ax^2 + bx + c \) and read off the \textbf{x-coordinate(s)} of the point(s) where they intersect.
- For example, you want to find the values of \( x \) for \( y = (x + 1)(x - 3) \), if:
  - \( y \) increases, as \( x \) increases:
    - \( y \) is positive:
      This will be the part of the graph above the \( x \)-axis, because above the \( x \)-axis the \( y \)-values of the graph are positive. So, the answer is \( x < -1 \) or \( x > 3 \).
  - The coordinates of the point at which \( y \) has its lowest value:
    - We already determined that the \( x \)-coordinate is 1.
    - Now, just substitute \( x = 1 \):
      \[
      y = (x + 1)(x - 3)
      = (1 + 1)(1 - 3)
      = (2)(-2)
      = -4
      \]
    - So, the point is \( (1, -4) \).

Sketch graphs

- A sketch graph of a straight-line graph must show these general features:
  - The intercepts with both axes if there are two intercepts. If the graph passes through the origin (the point \((0, 0)\)), it must show at least one other piece of information. We could show the gradient or a point on the graph.
  - A sketch graph is not a precise graph, but we must be able to determine the equation of the graph from the information shown on the sketch graph. Since the general equation of the graph is \( y = mx + c \), and \( m \) and \( c \) must be known, two pieces of information must be shown on the sketch graph.

- When the quadratic function is sketched, the following (with your students’ present knowledge) has to be shown:
  - whether the graph has a minimum \( \bigcup \) or a maximum \( \bigcap \).
  - the point where the graph intersects the \( y \)-axis.
  - the point(s) where the graph intersects the \( x \)-axis.

Inverse functions

The functions in the textbook are examples of graphs called hyperbolic graphs.

Explain to the class that these functions are discontinuous because they are not defined for all values of \( x \) or \( y \):

- In functions defined by an equation of the form \( y = \frac{1}{x + b} \):
  - \( y \) can never be equal to zero.
    - So, the graph of this function will never intersect the \( x \)-axis.
The values of \( y \) will become smaller and smaller but will never reach 0.
- We say that the \( x \)-axis or \( y = 0 \) is an asymptote of the graph. That means that it is a line to which the graph comes nearer to and nearer but never intersects.
- \( x \) can never be equal to \(-b\) (\( x \neq -b \)) because then one would divide by zero and division by zero is undefined.
- So, the graph of this function will never intersect the line \( x = -b \).
- That means that the line \( x = -b \) is a line to which the graph comes nearer and nearer but never intersects and also is an asymptote of the graph.

Relations and functions
Explain to the class what a function is by telling them that there are the following elements in a function:
- An independent variable (usually \( x \)). Its value is the input value (all the values that \( x \) may have is called the domain).
- A dependent variable (usually \( y \)) because its value depends on the value of the independent variable (\( x \)). Its value is also called the output value (all the values that \( y \) have, is called the range or the co-domain).
- A formula such as \( y = 2x + 3 \), which changes all the values of the independent variable (\( x \)) into the \( y \)-values that depend on the values of \( x \).
- For every value of the independent variable (usually \( x \)), there is only one value of the dependent variable (usually \( y \)).
- Examples of formulae that define functions: \( y = 2x + 3 \) (one-to-one function)
  \( y = 3x^2 - 2x - 5 \), because for each value of \( x \) (the independent variable), there is only one value of \( y \) (the dependent variable) (more-to-one function).
- An example of a formula that does not represent a function is \( y = \pm \sqrt{x} \), because if \( x = 9 \), for example, then \( y = 3 \) or \(-3\). So, for each value of the independent variable, there are two values of the dependent variable (a one-to-more relationship).

Areas of difficulty and common mistakes
Linear programming
- Students may find it difficult to work with the objective function as a family of parallel lines.
- They do not understand that to maximise they have to move the line as far as possible up along the \( y \)-axis while still keeping the same gradient. If the objective function is profit \( (P) \) in \( P = x + 4y \), for example, they need to make \( y \) the subject of the equation:
  \[
  -4y = x - P \\
  \therefore y = \frac{1}{4}x + \frac{1}{4}P
  \]
  So, because \( P \) is part of the \( y \)-intercept and we want the profit \( (P) \) to be as big as possible, the line with the gradient \(-\frac{1}{4}\) must be moved up as far as possible, while still satisfying the restrictions of the situation.
  So, part of the line must at least still touch one of the points of the lines that make up the boundaries of the region that contains the possible values of \( x \) and \( y \).
This region is called the feasible region.

• The opposite happens if one wants to minimise cost:
  For example, the cost, \( C = 20x + 10y \).
  Making \( y \) the subject, you get \( y = -2x + 0.1C \).
  Now the \( y \)-intercept (which is part of the cost) must be as low as possible while still touching at least one point that is part of the feasible region. The line in the middle below shows the optimal position of the objective function.
Quadratic graphs

- Some students find it difficult to read answers from graphs. If it is, however, explained very carefully, it should not be a problem.

The equation that defines the graph is \( y = 2x^2 + 5x + 2 \), for example:

- If you want to find the value of \( x \) (when \( y \) is a certain value):
  - You draw a horizontal line from that value on the \( y \)-axis until it intersects the graph.
  - Then you draw a vertical line from that point on the graph until the line intersects the \( x \)-axis and you read the answer from the \( x \)-axis.

- If you want to find the value of \( 2x^2 + 5x + 2 \), you look towards the \( y \)-axis for your answers.
  - If you want to find for which \( x \)-values \( 2x^2 + 5x + 2 > 0 \), for example, you read off the \( x \)-values of the graph for which the \( y \)-axis is positive.
  - The same is true for \( 2x^2 + 5x + 2 < 0 \) where you read the \( x \)-values of the graph for which the \( y \)-axis is negative.
  - If \( 2x^2 + 5x + 2 = 0 \), you read the \( x \)-values of the graph for which the \( y = 0 \), that is, where the graph intersects the \( x \)-axis.

- Some students may find it difficult to work out the correct \( y \)-values for their tables. Teach them to write their work out in their exercise books to prevent mistakes.

Relations and functions

Students may find it difficult to understand the notation \( f: x \rightarrow x^2 + 3x \).

- Always let them read it out loud: The function \( f \) maps \( x \) onto \( x^2 + 3x \).
- This means that each \( x \) is associated with a value obtained by \( x^2 + 3x \).
Learning objectives
By the end of this chapter, the students should be able to recall and know the following work:
1. Bar charts and pie charts.
2. Mean, median and mode.
3. Grouped data.
4. Range, mean deviation and standard deviation.
5. Probability.

Teaching and learning materials
Students: Textbook, an exercise book, writing materials and a calculator if possible.
Teacher: New General Mathematics 1, 2 and 3.

Teaching notes

Bar charts and pie charts
- Ungrouped data are usually graphically illustrated by bar charts and pie charts. There are spaces between the columns of the bar chart.
- The size of an angle in a pie chart is proportional to the frequency and the height of a column of a bar chart also is proportional to the frequency of the data.

Mean, median, mode
- The mean is calculated by adding all the data and dividing this total by the total number of data or the total frequency.
- Explain that the mode is the age, or shoe size or anything that occur most of the time or has the highest frequency.
- Explain the median as follows:
  - First numbers or measurement, for example, are arranged from the lowest to the highest.
  - If the number of measurements or numbers are odd, the middle measurement or number is the median.
  - If the number of measurement or numbers are even, then the median is the average of the two middle measurements or numbers.

Grouped data
- When there are a large number of values, these values are grouped in a frequency distribution table. In the table, the values are arranged in class intervals. If one class interval is 100–199, for example, then:
  - 100 and 199 are called the class limits.
  - Each class interval starts 0.5 below the lower class limit and ends 0.5 above the upper class limit and so 99.5 and 199.5 are called the class boundaries.
  - The class width is the difference between the upper and lower class boundaries, that is, 199.5 – 99.5 = 100, for example.
- The class mid-value or class mid-point is \( \frac{1}{2} \) the sum of the lower and upper class limits or in this example it is \( \frac{1}{2} (100 + 199) = 149.5 \).
- Grouped data are graphically illustrated by a:
  - Histogram that consists of rectangles with no spaces between them and of which the widths represent the class width and the height represents the frequency.
  - Frequency polygon where the top midpoints of each of the rectangles of the histogram are joined and which starts and ends with the midpoints of adjacent intervals on the horizontal axis.
- The modal class is the class with the highest frequency.
- Make sure that students know that for the ogive or cumulative frequency curve, they plot the upper class boundaries (on the horizontal axis) against the cumulative frequency (on the vertical axis).
- The accuracy of the readings from the ogive also depends on how accurate the curve is drawn.
Students can use a soft wire and bend it so that it passes through all the plotted points. They can then trace the curve of the wire on the graph paper to draw an accurate curve.

Range, mean deviation and standard deviation
- The range of data is the difference between the highest and the lowest data.
- For mean deviation and standard deviation, see Chapter 12 of this book.

Probability
- Probability is always given as a value from 0 to 1 or a value from 0% to 100% or as a value out of another value, for example the probability that a certain horse will win a certain race is 1 out of 4.
- Theoretical probability is probability based on logical reasoning written as \( \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}} \).
- Experimental probability uses experimental and/or statistical data to predict future events.

Addition and multiplication laws of probability
- Two successive events A and B are independent, if the outcome of the first event does not influence the outcome of the second event. For example:
  - You flip a coin and get tail. You then role a die and get a six.
  - It is clear that getting a tail does not influence getting a six if you role the die.
  - When events are independent, the sample size is not reduced.
  - Getting a tail does not reduce the number of choices for the second event.
- If two successive events A and B are independent then:
  - The probability that A and B happens \( = P(A \cap B) = P(A) \times P(B) \).
- Two successive events C and D are dependent, if the outcome of the first events influences the outcome of the second event. For example:
  - In your lunchbox you have 3 sandwiches and 2 apples.
  - You choose a food item and eat it. You then choose another food item and eat it.
  - Clearly the first event will influence the number of choices you have for the second event.
  - When events are dependent, the sample size is reduced.
  - Eating the first food item, reduces the number of choices you have choosing a second item.

Mutually exclusive events
- Mutually exclusive events can never occur simultaneously. For example, the event that a number is even and that the same number is odd are mutually exclusive, since a number can never be both even and odd.
- For mutually exclusive events this rule applies: \( P(A \cup B) \) (or the probability that either A or B happens) \( = P(A) + P(B) \). This is called the sum rule.

Areas of difficulty and common mistakes

Probability
- Words like “at most” and “at least” could give problems. Explain as follows:
  - Say you choose 3 cards from a pack of cards and you want to know what the probability is of choosing spades.
  - Then choosing at most 3 spades means that you could choose 3 spades, 2 spades, 1 spade or 0 spades.
  - Choosing at least 2 spades means that you can choose 2 spades or 3 spades.
- If a choice is made and the object is not put back, students tend to forget that the total number of objects is now 1 less and the number of the objects chosen and not put back is also 1 less. The quantity of the other kinds of objects stays the same.
  - For example, you have 3 black and 4 white balls in a bag.
  - If you take out one black ball and you do not put it back, there are 2 black balls left and the total number of balls now is 6.
  - The number of white balls remains the same.
  - So, the probability that a black ball is taken out again is \( \frac{2}{6} = \frac{1}{3} \) and the probability that a white ball is taken out at the second draw is \( \frac{4}{6} = \frac{2}{3} \).
Chapter 19: Geometry and vectors

Learning objectives
By the end of this chapter, the students should be able to recall and know the following work:
1. Angles.
2. Triangles.
3. Polygons.
4. Circles.
5. Geometric ratios.
6. Translation, rotation, reflection and enlargement.
7. Vectors.

Teaching and learning materials
Students: Textbook, an exercise book, writing materials and a calculator if possible.
Teacher: New General Mathematics 1, 2 and 3.

Teaching notes
* Emphasise that students must give a reason for each statement they make if the statement is the result of a theorem or an axiom.
To help them you could give them summaries of the theorems and suggestions of the reasons they can use.
Here are suggestions of how these summaries could look:

<table>
<thead>
<tr>
<th>Sketch</th>
<th>Theorem/axiom</th>
<th>Reason you must give if you use this theorem/axiom</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Sketch 1" /></td>
<td>If ABC is a straight line, ( \angle ABD + \angle DBC = 180^\circ ).</td>
<td>Sum ( \angle )'s on a str. line = 180°</td>
</tr>
<tr>
<td><img src="image2.png" alt="Sketch 2" /></td>
<td>If ( \angle ABD + \angle DBC = 180^\circ ), then A and C lie in a straight line.</td>
<td>The sum of adjacent angles = 180°</td>
</tr>
<tr>
<td><img src="image3.png" alt="Sketch 3" /></td>
<td>If two straight lines AB and CD intersect in O, ( \angle AOD = \angle BOC ) and ( \angle AOC = \angle DOB ).</td>
<td>Vert. opp. ( \angle )'s =</td>
</tr>
<tr>
<td><img src="image4.png" alt="Sketch 4" /></td>
<td>If AB \parallel DC and transversal GF intersects AB and DC, then alternate angles are equal.</td>
<td>Alt. ( \angle )'s =, AB \parallel DC</td>
</tr>
</tbody>
</table>
Sketch Theorem/axiom Reason you must give if you use this theorem/axiom

<table>
<thead>
<tr>
<th>Sketch</th>
<th>Theorem/axiom</th>
<th>Reason you must give if you use this theorem/axiom</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Sketch" /></td>
<td>If AB</td>
<td></td>
</tr>
<tr>
<td><img src="image2" alt="Sketch" /></td>
<td>If AB</td>
<td></td>
</tr>
</tbody>
</table>

In these last three theorems, you have to say that AB||DC, because:
1. Alternate ∠’s are not equal, if lines are not parallel.
   ![Sketch](image3)
2. Corresponding ∠’s are not equal, if lines are not parallel.
   ![Sketch](image4)
3. Interior opposite angles are not supplementary if lines are not parallel.
   ![Sketch](image5)

Triangles
- Triangles are congruent (≡), if they are identical in all respects.
- A triangle has 6 elements namely 3 angles and 3 sides.
  - If we want to construct a triangle, however, we need only 3 of the 6 elements.
  - The combinations of these three elements where no other triangle is possible are:
    - If three sides are given.
    - If two sides and the angle between them are given.
    - If two angles and a side is given. The side must be in the same position if all the triangles constructed using this information are to be congruent or identical.
    - If the hypotenuse, a side and 90° of a right-angled triangle is given.

Polygons
- When students find it difficult to recognise the exterior angle of a polygon, let them lengthen its sides and find as many as possible exterior angles.
- It may be a good idea for the students to recognise a rhombus and a rectangle as a special parallelogram that has all the properties of a parallelogram plus some other special properties.
  - A square could be seen as a special rectangle with some additional special properties or as a special rhombus with some additional special properties.
  - You could then ask the students to take these facts into consideration and give definitions for a rectangle, a rhombus and a square.

Circles
Here are suggestions of how the circle theorems could be summarised to help students to give reasons when they solve geometrical problems
1. Theorem 10 and the rules that follow can be remembered as:
Three facts:
- Centre circle
- Midpoint chord
- Perpendicular chord
If two of these facts are present, then the other one is also present.

a) Given: Centre circle, centre chord $\rightarrow \perp$ chord.

Reason given if this fact is used:
**OD bisects AB.**

b) Given: Centre circle, $\perp$ chord $\rightarrow$ centre chord.

Reason given if this fact is used: **OD $\perp$ AB.**

c) Given: Centre chord, $\perp$ chord $\rightarrow$ centre circle.

Reason given if this fact is used:
**AD = DB, MD $\perp$ AB.**

2. Angle at the centre of a circle subtended by a certain arc is equal to twice the angle at the circumference subtended by the same arc.

Reason given if this theorem is used:
**$\angle$ at centre $= 2 \times \angle$ at circumf. on AB.**
(Note that it is always easier to write the reasons as short as possible.)
3. Deductions from this theorem:
   a) ∠ in semi-circle.

4. Cyclic quadrilateral theorems:
   a) opp. ∠s in cyclic quad suppl.

5. Tangent theorems:
   a) Radius ⊥ tangent.
b)\[\begin{align*}
\text{Tangent} & \quad \text{Tangent}
\end{align*}\]

Reason given if this theorem is used: **Tangents from same point P are equal.**

c)\[\begin{align*}
\text{Reason given if this theorem is used: Angle between BD and chord BA = angle on BA.}
\end{align*}\]

**Geometric ratios**
- In all figures other than triangles, both the corresponding angles must be equal and corresponding sides in the same ratio for the figures to be similar.
- For triangles the following rules apply:
  - If corresponding angles are equal, corresponding sides are in the same ratio.
  - If corresponding sides are in the same ratio, corresponding angles are equal.

**Translation, rotation, reflection and enlargement**
- When you want to explain how the angle of rotation is determined you could tell the class the following:
  - Join A of the original figure to K the centre of rotation.
  - Join P the corresponding point of the image to K the centre of rotation.
  - Measure angle PKA.
  - In this case, $\angle PKA = 90^\circ$ and the rotational direction from P to A is anti-clockwise.

To find the centre of enlargement can be tricky if students do not know what the images of the specific points are, because to find the centre of enlargement a point must be joined to its image and another point must be joined to its image.
- The centre of enlargement then is the point where these two line segments intersect.
• Usually it can be seen from the form of the figure what the images of the different points are, otherwise it must be given.
• The scale factor is always measured from the centre of enlargement.
  • If the scale factor is 3 and the centre of enlargement is P, then \( \frac{PB'}{PB} = \frac{3}{1} \).
  • You should draw PB’ three times as long as PB (BB’ = 2PB).

![Diagram of a figure with points labeled P, B, B', and labeled coordinates]

• If the origin is the centre of rotation, these rules can make things much easier:
  • \( 90^\circ \) anticlockwise or \( 270^\circ \) clockwise: \((x, y) \rightarrow (-x, y)\)
  • \( 90^\circ \) clockwise or \( 270^\circ \) anticlockwise: \((x, y) \rightarrow (x, -y)\)
  • \( 180^\circ \) clockwise or anticlockwise: \((x, y) \rightarrow (-x, -y)\)
• There are also rules for reflection around certain lines that will make things easier:
  • Reflection in the \( x \)-axis: \((x, y) \rightarrow (x, -y)\)
  • Reflection in the \( y \)-axis: \((x, y) \rightarrow (-x, y)\)
  • Reflection in the line \( y = x \): \((x, y) \rightarrow (y, x)\)
  • Reflection in the line \( y = -x \): \((x, y) \rightarrow (-y, x)\)
• If the origin is the centre of enlargement, this rule applies:
  • If the sides of the enlarged triangle are \( k \) times longer than those of the original triangle: \((x, y) \rightarrow (kx, ky)\)

**Vectors**

• Students must remember that, if they represent a vector \((\vec{x}, \vec{y})\) on squared paper or graph paper:
  • A positive \( x \)-component is a horizontal displacement to the right (\( \rightarrow \)).
  • A negative \( x \)-component is a horizontal displacement to the left (\( \leftarrow \)).
• A positive \( y \)-component is a vertical displacement upwards (\( \uparrow \)).
• A negative \( y \)-component is a vertical displacement downwards (\( \downarrow \)).
• Remember that, if the direction of a vector is measured, it is always measured anti-clockwise from the horizontal.
• If the addition of vectors is drawn on graph paper, then the end point of one vector is joined to the starting point of the other one. The result or answer is then the vector that joins the starting point of the first vector to the end point of the last vector.

**Areas of difficulty and common mistakes**

• Students do not write reasons for their statements. Make a summary of reasons for them as shown above and insist on reasons.
• Students tend to leave out steps of their reasoning. Tell them that they are actually explaining the solution to somebody else and want to convince this person of their solution as well as make sure that the person fully understands the solution to the problem.
• Students tend to write the letters of two congruent triangles in the wrong order.
  • Emphasise that they have to write the letters according to the corresponding sides and angles that are equal.
  • Let them trace the letters according to the equal parts with their fingers and say the letters out loud.
• Students tend to say that two triangles are congruent if \( \text{AAS of one triangle are equal to AAS of the other triangle when the equal sides are not in the same position.} \)
  • If necessary, let them construct a pair of triangles where the equal side is not in the same position.
• Students find it difficult to write out a proof or a how they calculate something in a geometrical figure.
  • The only cure for this is practise and explanation and more practise and explanation.
  • It also helps to let the student first verbally explain the solution or proof to you and then write out the whole proof or calculation in essay format.
  • After this, the student can “translate” their essay in the formal form of a proof as illustrated in the textbook.
Geometric ratios

- Writing ratios in the correct order. Emphasise for example:

\[
\begin{align*}
\frac{AD}{DB} &= \frac{AE}{EC} \quad \text{and} \quad \frac{DB}{AD} = \frac{EC}{AE} \\
\frac{AD}{AB} &= \frac{AE}{AC} \quad \text{and} \quad \frac{AB}{AD} = \frac{AC}{AE}
\end{align*}
\]

- The order of the ratio of the one side is also valid for the other side.

- Writing the correct ratios of corresponding sides of two similar triangles. Help students as follows:

1. Write the letters of the two similar triangles in the order of corresponding angles that are equal.

\[
\triangle ADE \parallel \triangle ACB \quad \text{(the \parallel sign means similar)}
\]

2. Now write the ratios:

\[
\frac{AD}{AC} = \frac{DE}{CB} = \frac{AE}{AB}
\]
Learning objectives
By the end of this chapter, the students should be able to recall and know the following work:
1. Perimeter.
2. Area of plane shapes.
3. Surface area of and volume of solids.
4. Areas and volumes of similar shapes.
5. Latitude and longitude.

Teaching and learning materials
Students: Textbook, an exercise book, writing materials and a calculator, if possible.
Teacher: New General Mathematics 1, 2 and 3.

Teaching notes
Perimeter
When calculating the perimeter of a sector, emphasise that the students do not forget to also add two times the length of the radius to the part of the perimeter of the circle.

Area of plane shapes
• If in a triangle two sides and the angle between the two sides are given, the area formula is used. It states that area of a triangle = \( \frac{1}{2} \) side \( \times \) side \( \times \) sin (\( \angle \) between the two sides).
• Remember that the diagonal of a parallelogram bisects its area.
• The area of triangles or parallelograms on the same base or on equal bases and between the same parallel lines are equal.
• The area of a triangle is \( \frac{1}{2} \) the area of a parallelogram on the same base and between the same two parallel lines and the area of a parallelogram is equal to twice the area of a triangle on the same base and between the same parallel lines.

Surface area of and volume of solids
• If you cut through a solid parallel to its base and the cross section is identical or congruent to the base, the solid is a prism.
• Or: A prism is a solid object with two identical ends and flat sides:
  - The sides are parallelograms (4-sided shapes with opposite sides parallel).
  - The cross section is the same all along its length.
  - The shape of the ends gives the prism its name, such as “triangular prism”.
• The total surface area of a prism can also be calculated as follows, if we look at the net of the figure:

A cuboid:

\[
\text{Total surface area} = 2(l \times b) + 2(l + b) \times h
\]
**A cylinder:**

\[ r \hspace{1cm} \pi r \hspace{1cm} r \]

Total surface area = \(2\pi r^2 + 2\pi rH\)

**A triangular prism:**

\[ a \hspace{1cm} b \hspace{1cm} c \hspace{1cm} H \]

Total surface area = \(2\left(\frac{1}{2}bc\right) + \left(c + b + a\right)H\)

In general:

The total surface area for a prism = \(2 \times \text{base area} + \text{perimeter base} \times \text{height of prism}\)

**Areas and volumes of similar shapes**

See Chapter 4 of this book.

**Latitude and longitude**

See Chapter 7 of this book.

**Areas of difficulty and common mistakes**

**Surface area and volume of solids**

- Students may find it difficult to work out the total surface area or volume of a prism, if it is of another form than the four basic prisms shown above.
  The reason for this is that they cannot identify the bases of the prisms.
  They could overcome this difficulty if they imagine that they have a knife and that they cut through the solid like they would cut through bread to obtain identical slices.
  In this way, they could identify the base and apply the general formula for working out the volume or the total surface area of the solid.
- When working with composite solids, students may find it difficult to visualise the 3D forms.
  Their work can be made less complicated by drawing the separate solids and then working out the required volumes or areas.
- When calculating the volume of the frustum of a cone or a pyramid, students may find it difficult to identify the two similar triangles that are necessary to calculate the altitude of the remainder of the cone or pyramid.
  Teach them to always start with the length they want to work out.
  This length is part of a right-angled triangle of which the length of another side is known.
  Then they must look for another right-angled triangle of which the lengths of the corresponding sides are both known.

- Remind the students to always use the same units when they calculate the volume or total surface area of a solid.
- To avoid working with fractions they should always convert all units to the smallest unit.
- If the units, for example, are given as cm and m, they should convert all units to cm.
- If the volume is required to be in m³ or the total surface area is required to be in m², for example, convert the lengths to m from the beginning. In this way, errors are avoided, because it is much more difficult to convert cm³ or cm² to m³ and m² than it is to convert cm to m.

Areas and volumes of similar shapes

See Chapter 4 of this book.

Latitude and longitude

See Chapter 7 of this book.

Areas of difficulty and common mistakes

Surface area and volume of solids

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  This length is part of a right-angled triangle of which the length of another side is known.
  Then they must look for another right-angled triangle of which the lengths of the corresponding sides are both known.
Learning objectives
By the end of this chapter, the students should be able to recall and know the following work:
1. Pythagoras’s theorem.
2. Sine, cosine and tangent ratios.
3. Angles of 45°, 60°, 30°, 0° and 90°.
4. The sine rule.
5. The cosine rule.
6. Angles between 0° and 360°.

Teaching and learning materials
Students: Textbook, an exercise book, writing materials and a calculator if possible.
Teacher: General Mathematics 1, 2 and 3.

Glossary of terms
Hypotenuse is the side opposite the 90° angle in a right-angled triangle.
Right angle is an angle of size 90°
Pythagoras was a Greek philosopher who lived about 2 500 years ago.
Pythagoras’ theorem states that in any right-angled triangle the square of the side opposite the right angle is equal to the squares of the other two sides added.
Pythagorean triple is a set of three whole numbers that give the lengths of the sides of a right-angled triangle.

Teaching notes
Pythagoras’s theorem
The side opposite the 90° angle in a right-angled triangle is always the hypotenuse, even if the triangle is in a different orientation from what is expected.

Sine, cosine and tangent ratios
• When students solve a problem where they have to work out the sides of the triangle, they should reason as follows:
  ▪ Which side do I want to calculate? Which side do I have? Write down the ratio as the side that I want \( \frac{1}{\sin} \) the side that I have \( \frac{f}{2} \).
  ▪ Ask yourself what trigonometric ratio of the angle in the problem does this give? It could be a trigonometric ratio or \( \frac{1}{\tan} \) the trig ratio \( \frac{f}{2} \).
  ▪ Write \( \frac{1}{\sin} \) and solve for the side that you want.
  ▪ Do not work out the answer if it is not the final answer. Use the expression as it is, to find the final answer in the next step.
• Note that sin\(^{-1}\) on the calculator, does not mean \( \frac{1}{\sin} \) (which makes no sense in any case, because one must have the sin of a specific angle), but it means that this gives us the angle of which that ratio is the sin-ratio. The same applies for the cos-ratio and the tan-ratio of an angle.
• When students have to solve a problem where they have to work out angles, they must look at the sides given in the right-angled triangle and ask themselves what trigonometric ratio it gives of the angle they want to calculate. If the angle is \( \theta \), they should then write (trig. ratio) \( \theta = \frac{\text{side}}{\text{other side}} \).
• If the answer of the next section of the problem depends on the answer of the previous section, it is the most accurate if you use the expression that gave the answer of the previous section.
Angles of $45^\circ$, $60^\circ$, $30^\circ$, $0^\circ$ and $90^\circ$

- Use the sketches in the textbook to explain the sin, cos and tan ratios of these three angles.
- Students should remember the $45^\circ$ triangle.
- For $60^\circ$ and $30^\circ$ they only have to remember one triangle:

![60° 30° √3/2 1](image)

- The trigonometrical ratios of $0^\circ$ and $90^\circ$, $180^\circ$, $270^\circ$ and $360^\circ$ can be easily done, if students remember the unit circle as shown below.

![unit circle](image)

The sine rule

- Students have to remember that, if there is a side and an angle opposite each other, they can use the sine rule.
- To make using the sine rule easy:
  - Use this version of the sine rule if you have to work out a side:
    
    \[
    \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
    \]
  - Use this version of the sine rule if you have to work out an angle:
    
    \[
    \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
    \]
  - Remember the angle could also be an obtuse angle, because the sines of obtuse angles are positive.
  - When you are given side, side, angle of a triangle that you have to solve, the possibility exists that there are two possible triangles. To find out whether two triangles are possible, follow this procedure:
    1. Draw a sketch of the given triangle showing all the information given.
  2. If the side opposite the given angle is equal or longer than the side adjacent to the given angle, there is only one possible triangle.
  3. If the side opposite the given angle is shorter than the side adjacent to the given angle, there are two possible triangles.
  4. If the given angle is obtuse, the side opposite this angle is obviously longer than the side adjacent to the given angle. The longest side of a triangle is always opposite the largest angle of the triangle. In a triangle, there is only one obtuse angle possible, because the sum of the angles of a triangle is equal to $180^\circ$.

The cosine rule

- Students should use the cosine rule, if in a triangle there are no sides and angles opposite each other. If two sides with the angle between them are given, then they can work out the side opposite the angle by applying the cosine rule as follows:
  
  \[(\text{Opposite side})^2 = (\text{adjacent side})^2 + (\text{other adjacent side})^2 - 2(\text{adjacent side})(\text{other adjacent side})(\cos \text{ of } \angle \text{ between the two sides})\]

  Now tell them that they now have (opposite side)$^2$ and they first have to get the square root of their answer to get the length of the side.

  So, in the beginning students would for example have written $b^2 = \ldots$, but for the final answer they should not forget to now write $b = \ldots$ when they get the square root of the first answer.

- If three sides of a triangle are given, the cosine rule is always used, but students often do not know how to apply the cosine rule here. Teach them to do the following:
  1. Choose an angle.
  2. Then write: $\cos$ of that angle
    
    \[
    \cos \theta = \frac{(\text{adjacent side})^2 + (\text{other adjacent side})^2 - (\text{opposite side})^2}{2 \times (\text{adjacent side})(\text{other adjacent side})}
    \]

    - In all these calculations, students must make sure that their calculator is set on degrees and not on radians or grads.

- After a side of a triangle was been worked out using the cosine rule, an angle must be worked out using the sine rule.
  1. In this case a side and a side and an angle not between the two sides are given.
  2. This combination of measurements of a triangle may result in the possibility of two triangles, if the side opposite the given angle is shorter than the side adjacent to the given angle.
• To prevent this happening and the students becoming confused, teach them to always work out the angle opposite the shortest side first.

Angles between 0° and 360°
For complete notes about this section, see Chapter 6 of this book.

Areas of difficulty and common mistakes
• When solving problems involving bearings and distances, students may find it difficult to make the sketches. Teach them to always start with the N–S and W–E cross and to work from there.
• Students do not realise the difference between working out the hypotenuse and one of the other two sides of a right-angled triangle using Pythagoras’s theorem. To prevent this the lengths of the sides should be substituted in the theorem and then you solve for the unknown side. For example:

\[ 8^2 + a^2 = 10^2 \]
\[ a^2 = 10^2 - 8^2 \]
\[ = 100 - 64 \]
\[ = 36 \]

Only when students know the work very well could they immediately write \( a^2 = 100 - 64 \) (always the square of the longest side minus the square of the other given side).

• Writing the sine, cosine and tangent ratios with capital letters:
They should write sin A and not Sin A, cos A and not Cos A, tan A and not Tan A.

• Students tend to write sin or cos or tan without associating these ratios with an angle. Emphasise that writing sin, cos or tan only, has no meaning.

• In a problem the student may write sin \( \theta = 8,866 \) = 60° instead of \( \therefore \theta = 60° \). You can only prevent a problem like this, if you look at the work of the students and tell them repeatedly what is correct.

• Students do not always see sin of an angle as a ratio. They would give senseless answers such as: \( \sin 32^{\circ} = 1 \) or \( \cos 2A = 2\cos A \). The only way you can correct this thinking mistake is to let them work out the values using tables or scientific calculators.

• Students still get the square roots of separate terms and do not realise that they first have to add or subtract the terms and then get the square root of the answer. For example, \( \sqrt{100 - 36} = \sqrt{64} = 8 \) and not \( \sqrt{100} - \sqrt{36} = 10 - 6 = 4 \).
Learning objectives
By the end of this chapter, the students should be able to recall and know the following work:
1. Constructions.
2. Loci.

Teaching and learning materials
Students: Textbook, an exercise book, writing materials and a calculator if possible.
Teacher: New General Mathematics 1, 2 and 3, and geometric instruments.

Teaching notes
Constructions
• Use your board compass and straight edge to again illustrate all the constructions on pages 234–235.
  ▪ Let the students do the same constructions on plain paper while you are illustrating them on the board.
  ▪ Now, also do Examples 1–3 on pages 235–236 on the board and let the students do the same constructions on plain paper while you are illustrating on the chalkboard how they should be done.
• When you do the examples on the chalkboard:
  ▪ Emphasise that it is essential that a sketch with all the information must be made before the construction is attempted.
  ▪ Insist that everybody uses a sharp pencil.
  ▪ Insist that all construction arcs are shown or visible.
  ▪ Let students measure the lengths of lines with a compass (ideally a measuring compass) and then putting the compass on a ruler and reading off the length on the ruler, because measuring with a ruler only may be inaccurate if a parallax error is made.
  ▪ Insist that every student work with you and do the same constructions while you illustrate them on the chalkboard.

Loci
• When you explain specific loci, describe the condition of the locus and then first ask the students to describe what the locus would look like.
  ▪ To be able to do combinations of loci the students should know the basic loci by heart. They should for example, know that the locus of points:
    ▪ Equidistant from two fixed points A and B is the perpendicular bisector of AB.
    ▪ Equidistant from two fixed lines AB and AC is the bisector of ∠BAC.
    ▪ Equidistant from a fixed point O is a circle with O as centre.
    ▪ Equidistant from two lines AB and CD that intersect in O, are the bisectors of the angles formed at O.
    ▪ There are, say 30 mm from a line AB, and two lines on both sides AB, parallel to AB and 30 mm from AB.
    ▪ Equidistant from two parallel lines AB and CD, is a third line parallel to AB and CD and precisely in the middle of AB and CD.

Areas of difficulty and common mistakes
• Students use blunt pencils. Insist that they sharpen their pencils.
  If necessary, provide a sharpener.
  Also insist that the pencil has a point that is not too soft and also not too hard.
  An HB point is the best.
• Students do not show construction arcs. Insist that they do.
• The construction of loci can be very difficult as the concept may be very difficult to understand. This can be overcome if:
  ▪ Students know all the basic loci.
  ▪ Make a rough sketch before attempting to make the accurate construction.
  ▪ If you teach students to read very carefully through the instructions of the problem and to make the rough sketch step by step as the reading progresses.